[PS]A Spectral Approach to Conditional Simulation*

Ilnur Minniakhmetov[1], Anna Pergament[1], and Alina Akhmetsafina[1]

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[1]Moscow Institute of Physics and Technologies (delnuro@gmail.com)

Abstract

The generation of 2D and 3D grids of normally distributed random fields conditioned on well data is often required in reservoir modeling. Such fields can be obtained from the application of three groups of methods: unconditional simulation with kriging interpolation (turning band or spectral methods), sequential gaussian simulation (SGS), and Cholesky factorization of the covariance matrix. However, all methods have limitations. First, it is shown, that the second moment of the process conditionally simulated with the help of the kriging method are not identical to the target second moment (a priori known statistics). Second, SGS cannot be calculated without limitation on a number of neighbors. As a result, SGS is only asymptotically exact. Third, which has the advantage of being general and exact, is to use a Cholesky factorization of the covariance matrix representing grid points correlation. However, for fields of large size, the Cholesky factorization can be computationally prohibitive. Another approach is to use spectral function of full covariance matrix. In this work, we show that covariance of two arbitrary spectral components of conditional process could be represented as the product of functions. In this case, the Cholesky factorization could be considerably simplified. A feature of this approach is its computational simplicity and suitability to parallel implementation.

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A Spectral Approach to Conditional Simulation

Ilnur Minniakhmetov, Anna Pergament, Alina Akhmetsafina

Moscow Institute of Physics and Technologies

minnix@yandex.ru

Abstract

Generation of 2D and 3D normally distributed random fields conditioned on well data is often required in reservoir modeling. Such fields can be obtained by using three groups of methods: unconditional simulation with kriging interpolation (turning band or spectral methods), Sequential Gaussian Simulation (SGS) an d Cholesky factorization of the covariance matrix. However, all these methods have limitations. First, it is known, that the second moment of the stochastic process conditionally simulated by kriging method is not identical to the target second moment (a priori known statistics). Second, SGS can't be calculated without limitation on the number of neighbors. As a result, SGS is only asymptotically exact. Third approach, which has the advantage of being general and exact, is to use a Cholesky factorization of the covariance matrix representing grid points correlation. However, for the large fields the Cholesky factorization can be computationally expensive. In this work we present an alternative approach, based on the usage of spectral representation of a conditional process. It is shown that covariance of two arbitrary spectral components could be factorized into functions of corresponding harmonics. In this case the Cholesky decomposition could be considerably simplified. The advantage of the presented approach is its accuracy and computational simplicity.

1. Introduction

Mathematical modeling is often required in field development optimization problems, i.e. an optimal control of waterflooding, selection of enhanced oil recovery methods, design and execution of hydraulic fracturing and other workovers programs. The main stages of oil-gas reservoir modeling are

- Geological modeling
- Upscaling
- Hydrodynamic modeling

The geological model consists of wells, a three-dimensional grid, lithotofacial and petrophysical (porosity, permeability, oil and gas saturations) fields. The petrophysical properties are assumed to be stationary stochastic processes (Fig. 1) within each lithofacie. There are two approaches to construct these processes: the statistical estimation and generation of realizations conditioned on well data, i.e. the conditional simulation.

First one could be performed using different interpolation methods [3, 1]. The classical interpolation techniques mentioned above are effective for sufficient smooth functions, while the realizations of stationary random fields are likely to belong to L_2 space [7, 5]. To overcome the limitations of classical approaches, the krigging algorithm (Best Linear Unbiased Estimation) was developed [4, 8, 9].

In this paper we consider the second approach, i.e. the conditional simulation in geological modeling, and its numerically efficient realization. We present the alternative approach of conditional simulation based on Cholesky decomposition of the covariance matrix in Fourier domain.

2. Problem statement

Let $\vec{X}$ (Fig. 1) be a stationary random vector with zero mean and covariance matrix $C = E(\vec{X}\vec{X}^+)$. Suppose the values of random vector $\vec{X}$ be known at m well-crossed cells of regularly spaced grid

$$X_{n_1} = y_1, X_{n_2} = y_2, \dots X_{n_m} = y_m, \tag{1}$$

where $n_1, n_2 ... n_m$ are cell indices.

The focus of this paper is generation of realizations of the vector $\vec{X}$ conditioned on well data (1).

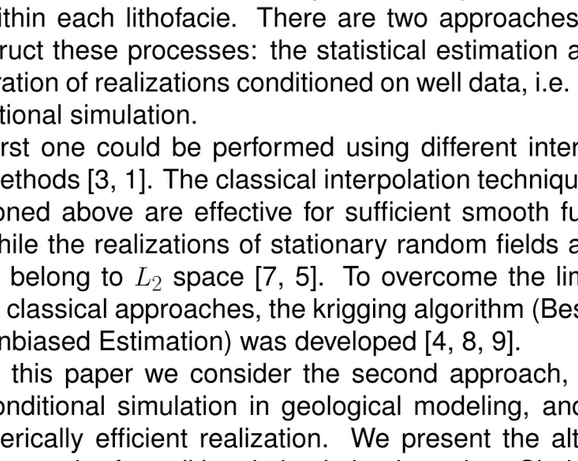

Figure 1: *Realizations of 1D stationary stochastic process*

3. Decomposition of covariance matrix in Fourier domain

It is known [2] that the covariance $\sum$ of process $\vec{X}$ conditioned on (1) is:

$$\sum = C - B^T C_m^{-1} B, \tag{2}$$

where:
C is unconditional covariance of $\vec{X}$
C_m is unconditional covariance of well data (1)
B is unconditional covariance of $\vec{X}$ and well data (1)

For Fourier transformation of the covariance matrix discrete Fourier transformation (DFT) is used. With help of matrix computation and theory of stochastic processes the covariance matrix $\sum$ in Fourier domain could be expressed as follows:

$$\begin{aligned} &\textstyle\sum^{\omega} = F \sum F^+ = D(2ND^{-1} - WC_m^{-1}W^+)D \\ &W_{kl} = e^{-\frac{2\pi i(k-1)(n_l-1)}{2N}}, k = 1..N, l = 1..m \end{aligned} \tag{3}$$

where:
F is Fourier transform matrix
D is a diagonal matrix with components of the vector $FC^{(1)}$
$C^{(1)}$ is the first column of the matrix C
C_m^{-1} is the inverse covariance matrix of components (1)
n_l is crossed cell
N is the total number of cells
m is the number well-crossed cells

It should be noted, that matrix $\sum^{\omega}$ is positive defined and Hermitian.

To calculate Cholesky factorization of the matrix $\sum^{\omega}$ it is sufficient to decompose the matrix

$$G = 2ND^{-1} - WC_m^{-1}W^+. \tag{4}$$

Following that:

$$\sum^{\omega} = DGD = DLL^+D = (DL)(DL)^+, \tag{5}$$

where L is the Cholesky decomposition of G.

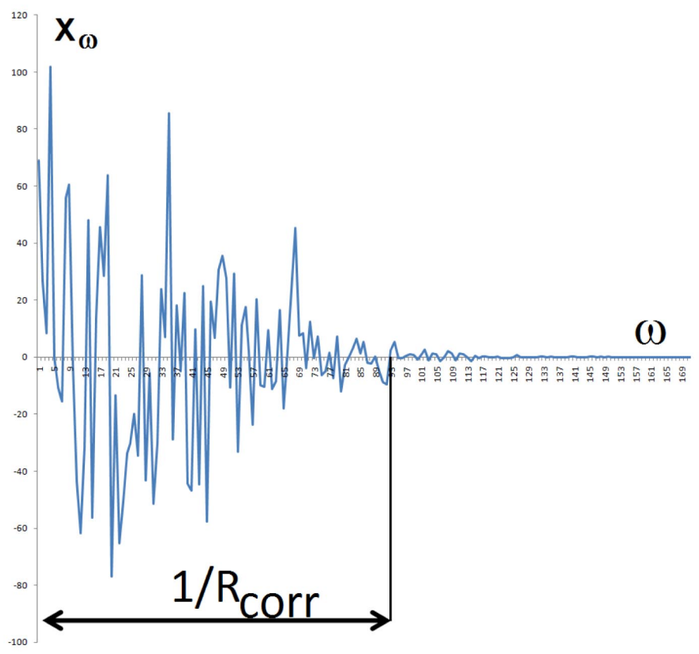

Figure 2: *Fourier transform of realization of 1D stationary stochastic process*

Now, if $\vec{x}$ is the realization of the random vector $\vec{X}$, the Fourier transform of $\vec{x}$ (Fig. 2) is

$$\vec{x}^{\omega} = F\vec{x} = DL\vec{\eta}, \tag{6}$$

where $\vec{\eta}$ is a column-vector of independent standard normal random variables.

Using mathematical induction it is easy to prove that Cholesky decomposition algorithm of G is reduced as follows:

For j=1..2N

$$\vec{w}_j = \left(W^+\right)^{(j)}$$
$$\vec{c} = \left(FC'^{(1)}\right)_j$$
$$L_{jj} = \sqrt{\frac{2N}{j} - \vec{w}_j^+ A^j \vec{w}_j}$$
$$L_{ij} = -\vec{w}_i^+ \vec{g}_j, i < j$$
$$\vec{g}_j = \frac{A^j \vec{w}_j}{L_{jj}}$$
$$A^{j+1} = A^j + \vec{g}_j \vec{g}_j^+$$

Algorithm 1: *Algorithm for the Cholesky decomposition of the covariance matrix in Fourier domain.*

With help of back DFT algorithm the realization of the random vector $\vec{X}$ has the form

$$\vec{x} = F^{-1}\vec{x}^{\omega} \tag{7}$$

4. Estimation of conditional simulation parameters and computational costs

One of the main characteristic of conditional simulation methods is their computational costs. Suppose m is the number of crossed cell, N is the total number of cells. According to Algorithm 1, there are $O(m^3)$ operations for the matrix inversion, $O(Nm^2)$ operations for calculation of matrix additions and matrix-vector multiplications, $O(N^2)$ operations for DFT (or $O(Nlog(N))$ in FFT case). Finally, computational cost of algorithm is $O(N^2 + m^3 + Nm^2)$ (or $O(Nlog(N) + m^3 + Nm^2)$ in FFT case) operations. Let us remark that in most geological models $m << N$, then computational burden is $O(N^2 + Nm^2)$.

For quality evaluation, the estimation of conditional simulation parameters, i.e. mathematical expectation, dispersion, and covariance should be calculated.

$$\begin{aligned} &\hat{\mu}_i = \hat{E}(X_i) = \tfrac{1}{N_r}\textstyle\sum_{k=1}^{N_r} x_i^k \\ &\hat{\sigma}_i^2 = \hat{D}(X_i) = \tfrac{1}{N_r}\textstyle\sum_{k=1}^{N_r}\left[(x_i^k - \mu_i)\right]^2 \\ &\hat{\textstyle\sum}_{ij} = \hat{C}ov(X_i X_j) = \tfrac{1}{N_r}\textstyle\sum_{k=1}^{N_r}(x_i^k - \mu_i)(x_j^k - \mu_j) \end{aligned} \tag{8}$$

where:
X_i is the component i of the random vector $\vec{X}$
x_i^k is the realization k of the component X_i
N_r is the number of realizations
μ_i is expectation of x_i

Conditional expectation and dispersion for an arbitrary cell is shown in Fig.3 and Fig.4 respectively.

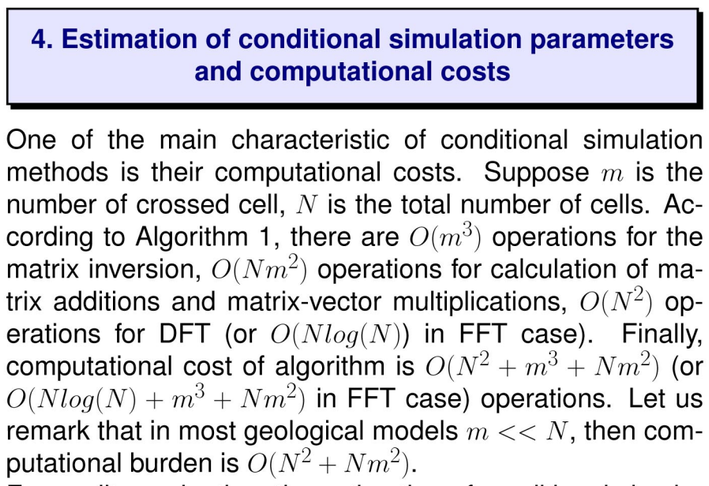

Figure 3: *The estimation of conditional mathematical expectation depends on number of realizations*

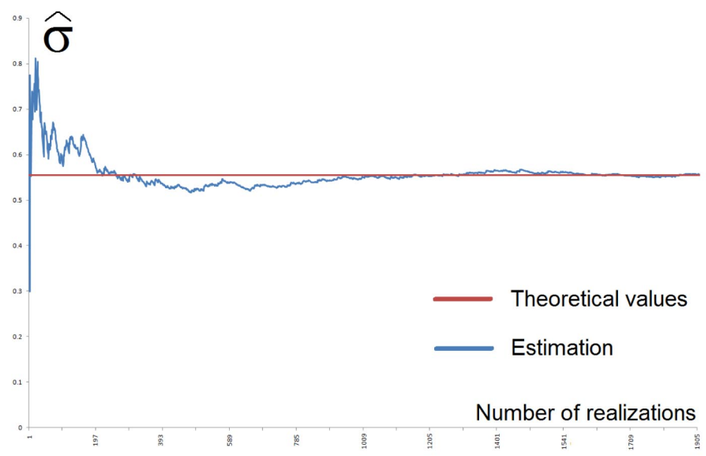

Figure 4: *The estimation of conditional dispersion depends on number of realizations*

Conditional covariance in an arbitrary cell in 1D case for 2000 realizations is shown in Fig.5.

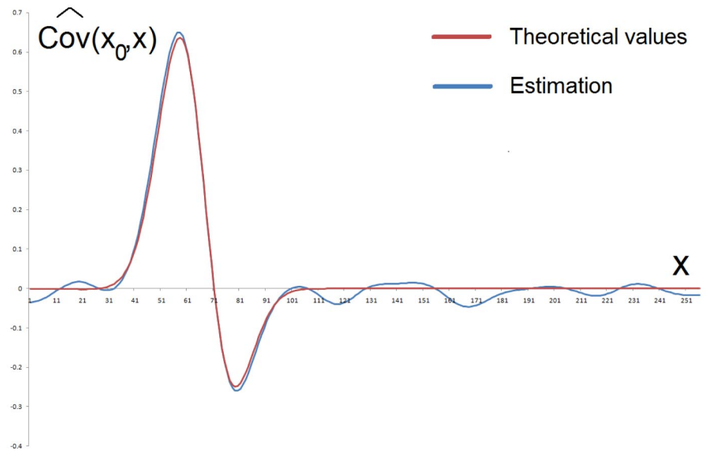

Figure 5: *The estimation of conditional covariance in an arbitrary cell for 2000 realizations*

Let's compare the estimations of spectral approach and SGS. Fig. 6 shows that the covariance of spectral approach realizations is close fitted to the theoretical covariance.

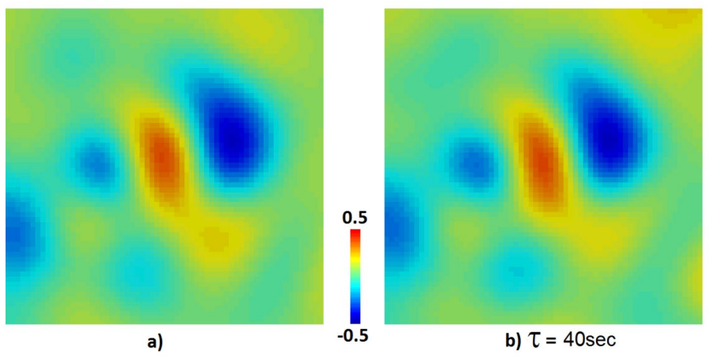

Figure 6: *Theoretical values and the estimation of conditional covariance for 1000 realizations in 2D case.*
a) - Theoretical values, b) - Estimation for spectral approach

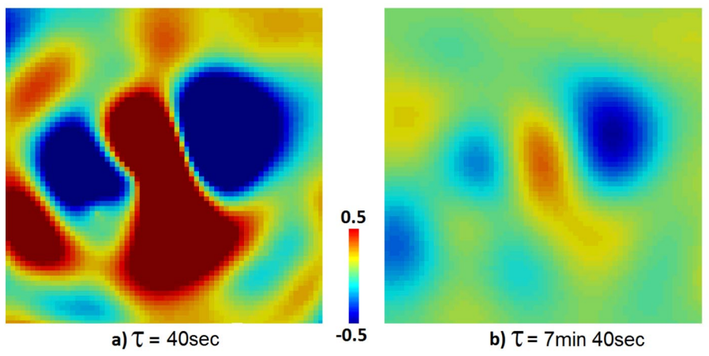

Figure 7: *The estimation of conditional covariance for 1000 realizations in 2D case.*
a) - Estimation for SGS with 12 neighbors, b) - Estimation for SGS with 50 neighbors

Fig. 7 shows that SGS approach gives a good results only with large number of neighbors, but in this case SGS can be computationally too expensive.

5. Examples

The example results of 2D and 3D conditional simulation of porosity are shown in Fig. 8, Fig. 9 respectively. The parameters of model:

- $\sim 500000(128 \times 128 \times 32)$ cells
- field square is $5000 \times 5000m^2$
- 4 wells
- Covariance function:

$$C(\vec{r},\vec{r}') = 0.06exp(-\sqrt{\frac{(x-x')^2}{R_x^2}+\frac{(y-y')^2}{R_y^2}+\frac{(z-z')^2}{R_z^2}})$$

where:
$\vec{r} = (x,y,z)$ and $\vec{r}' = (x',y',z')$ are coordinates of two arbitrary cells
$R_x = 1000m$ is correlation radius along the axis x
$R_y = 1000m$ is correlation radius along the axis y
$R_z = 3m$ is correlation radius along the axis z

- Calculation time of one realization is $\sim 5min$

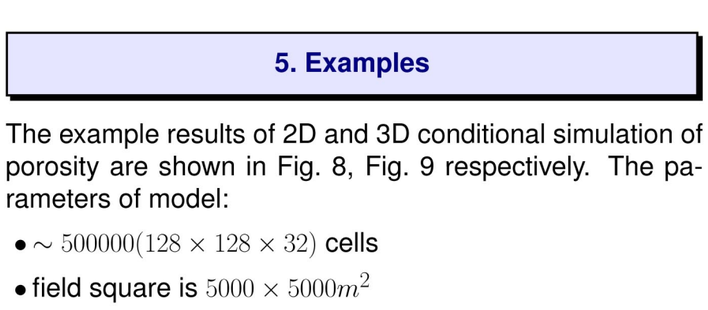

Figure 8: *Two realizations of conditional simulation using spectral approach in 2D case. Wells locations are depicted as points*

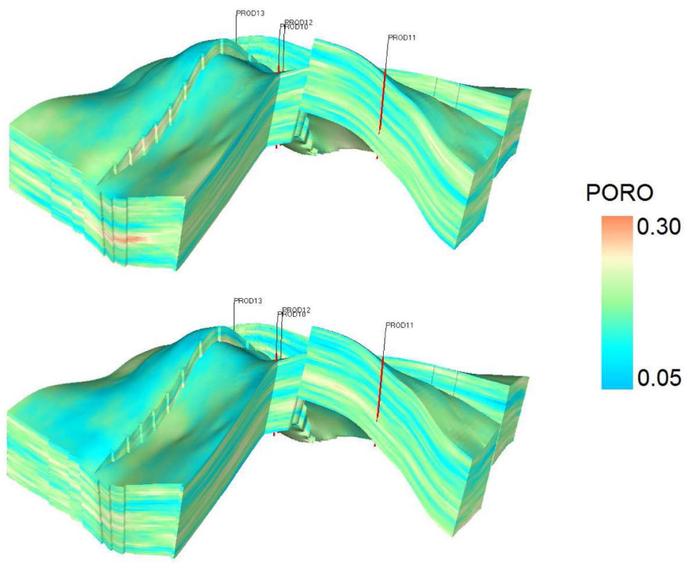

Figure 9: *Two realizations of conditional simulation using spectral approach in 3D*

6. History matching

Traditionally, reservoir development decisions are based on a production forecast from a single history matched reservoir. The main aspect of history matching, i.e. optimization procedure, is parametrization.
In this work we have concentrates on the spatial distribution of porosity and permeability witch can be parametrized in various ways:

- **Grid Block**.
  The approach is to consider all grid blocks. The main problems are the large number of parameters and the lack of spatial continuity.
- **Region**.
  The use of homogeneous regions is the way to reduce the number of parameters. The assumption of homogeneity within a region may not be justifiable and lead to abrupt changes between the boundary regions.
- **Pilot points**.
  A number of pilot points (p.p) are used to build smooth spatially correlated correction of porosity and permeability. The problems: difficult algorithm of p.p selection procedure and p.p change porosity and permeability fields locally.
- **Global parametrization**.
  A final class of parameters are those that cannot be linked to a particular spatial location, called global or underlying parameters: mean values, standard deviation, correlation length.

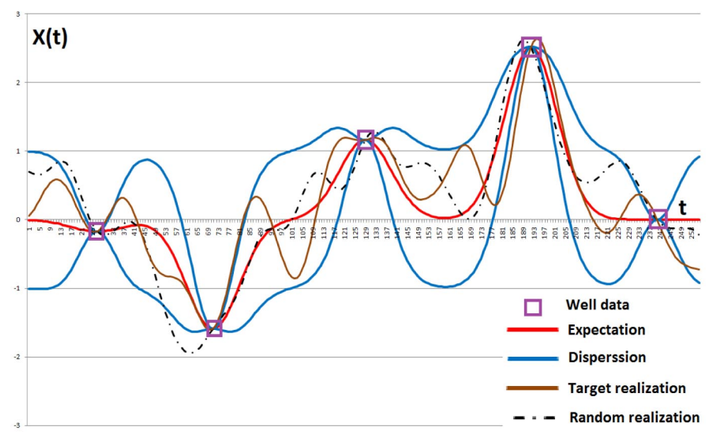

Figure 10: *History matching by porosity and permeability field parametrization*

Fig. 10 shows the main problem of parametrization techniques, i.e. it is necessary to transform a random realization to target realization with minimal number of parameters.
The approach was described in this paper to concern with the global parametrization technique. As the parametrization we use Fourier transform of stochastic process (6). Because of its rapidly decreasing form (Fig. 2) we could use the first several components of $\vec{\eta}$ as parameters. The main problem of this technique is conserving the distribution of parameters, i.e. the parameters should be independent standard normal random variables.
Through the Bayesian prior function this problem could be resolved. The general formula for the Bayesian posterior distribution is given by

$$f(p\,|o) = cf(p\,|o)f(p)$$

where:
$f(p\,|o)$ is the likelihood function
$f(p)$ is the prior function
When both the prior distribution on the parameters and the production uncertainty are assumed to be Gaussian, including the prior result in an extra sum of squares term ([6]),here denoted in vector notation,

$$f(p\,|o) = c\exp\left(-\left\{\Delta + (p-\mu_p)^T C_p^{-1}(p-\mu_p)\right\}\right)$$

where:
Δ is sum of squares of production and observation data difference
p is the vector of parameters
μ_p is the vector of expectations of p
C_p is the covariance matrix of p
In our case the parameters are independent and standard, then the covariance matrix is the identity matrix, expectation is zero, and

$$f(p\,|o) = c\exp\left(-\left\{\Delta + \sum(p_i)^2\right\}\right)$$

7. Conclusions

In this wok we have shown that the spectral approach has advantages:

1. **High accuracy in comparision with SGS**
2. **Releativly low compututional costs**
3. **Easy porosity and permeability parametrization in history matching problems**

Being relatively fast and accurate, spectral approach algorithm would be very useful in other application areas of conditional simulation: electronics, finance, game theory, etc. Further researches should be focused on the optimization and parallel implementations of the algorithm.

8. Acknowledgements

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