

Padé Approximation for Analyzing Multiple Reflections

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Summary

The paper considers rational Padé approximation of the z-transform function of a time-dependent minimum phase signal. We present a derivation of reflection and transmission coefficients in layered media which are related to infinite impulse response (IIR) filters as a rational function of a special form. The $[p, q]$ -Padé approximation of the z-transform function is formulated as a constrained least squares minimization problem with regularization constraints provided by the minimum phase signal. Numerical simulations for reconstruction of the z-transform function and its use as an inverse filter for a multiple model demonstrate the effectiveness of the presented approach.

Introduction

The numerical experiments described in this work were conceived as a test of how the technique of least squares Padé approximation could be used in a typical seismic data processing application. We consider rational Padé approximation of the z-transform function of a time-dependent minimum phase signal. We review the derivation of reflection multiples for a double interface, and observe the multiple signal can be modelled by an IIR filter of a simple form, with coefficients determined by the reflection and transmission parameters. The Padé method features a numerical approach that works directly on the data. There is no need to transform to the Fourier (or other) domain. We set up the Padé approximation problem using the seismic data directly, with some choice on the rational function form to reduce the dimension of the solution space. The rational ($[p, q]$ -Padé) approximation of the z-transform function is formulated as a constrained least squares minimization problem with regularization constraints provided by the minimum phase signal. Results of some numerical experiments in building the Padé approximating filter, and its use as an inverse filter to remove the multiples demonstrate the effectiveness of the presented approach.

Multiple Reflections and Transmissions

With an incoming wave $e^{i(\omega t + k_1 x)}$ on the right of the interface at $x = 0$ (refer to the left Fig.1), the transmitted and reflected waves are generated as $T e^{i(\omega t + k_2 x)}$ and $R e^{i(\omega t - k_1 x)}$, respectively. Using the continuity of the total

waveforms and the normal derivatives at $x = 0$, the reflection and transmission coefficients are derived as $R_{\leftarrow} = (1 - r)/(1 + r)$, $T_{\leftarrow} = 2/(1 + r)$, and $r = k_2/k_1$ is the relative index of refraction with the wave numbers k_1 (right medium) and k_2 (left medium). If we reverse directions, then we have $R_{\rightarrow} = -R_{\leftarrow}$, $T_{\rightarrow} = rT_{\leftarrow}$. For two interfaces (right Fig. 1), there will be multiple internal reflections. The total reflectivity is derived as $R_{total} = R_{\leftarrow}^1 - T_{\rightarrow}^1 (R_{\rightarrow}^1)^{-1} T_{\leftarrow}^1 + T_{\rightarrow}^1 (R_{\rightarrow}^1)^{-1} (1 - R_{\rightarrow}^1 D R_{\leftarrow}^2 D)^{-1} T_{\leftarrow}^1 = a + b(1 - cD^2)^{-1}$ for some constants a, b, c , where $R_1 = R_{\leftarrow}^1$, $R_2 = T_{\rightarrow}^1 D R_{\leftarrow}^2 D T_{\leftarrow}^1$, $R_3 = T_{\rightarrow}^1 D R_{\leftarrow}^2 D [R_{\rightarrow}^1 D R_{\leftarrow}^2 D] T_{\leftarrow}^1$. Similarly, the effective transmission T_{total} can be derived analytically as $T_{total} = T_{\leftarrow}^2 D (1 - R_{\rightarrow}^1 D R_{\leftarrow}^2 D)^{-1} T_{\leftarrow}^1 = b' D (1 - cD^2)^{-1}$ for constants b', c .

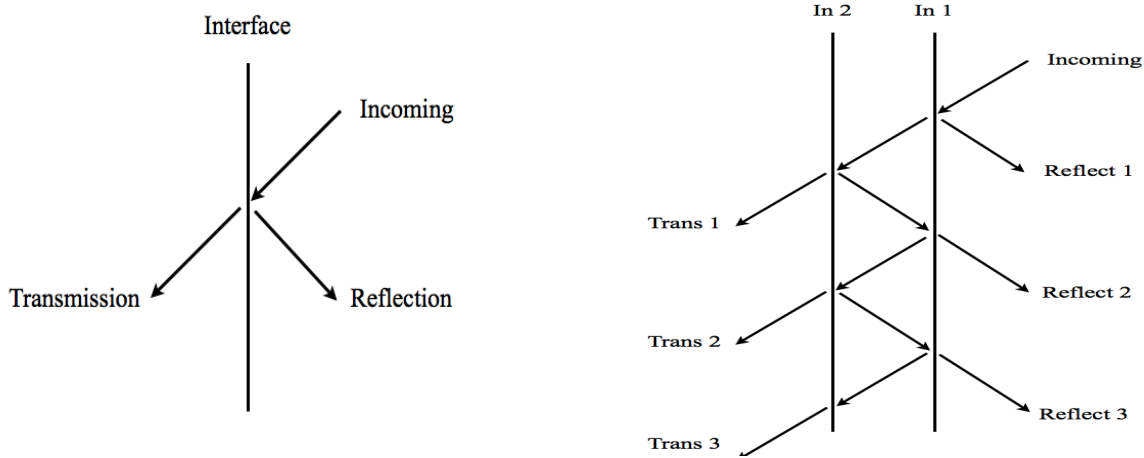


Figure 1: A reflection and transmission event across a single interface at $x = 0$ (left) and two interfaces (right).

Modeling with IIR Filters and Padé Approximation

The responses R_{total} and T_{total} can be modelled with IIR filters of very similar form as rational functions $G(z) = a + b/(1 - cz^{2d}) = (\alpha + \beta z^{2d})/(1 + \eta z^{2d})$, $H(z) = b' z^d/(1 - cz^{2d})$, where $\alpha = a + b$, $\beta = -ac$ and $\eta = -c$ are related to the multiple reflectivities, and d is an integer that models the delay of the signal through the gap. The filters $G(z)$ and $H(z)$ are stable since $c = R_{\rightarrow}^1 R_{\leftarrow}^2 < 1$. The integer d can be computed from the width of the gap, the velocity of sound in the gap, and the sample rate of the sampled signal given by $d = \text{sample rate} * \text{length}/\text{velocity}$.

The developed numerical inversion method for constructing Padé approximation of $G(z)$ is given as $G(z) \cong G_{[p,q]}(z) = a(z)/b(z)$ where $a(z) = a_0 + a_1 z + \dots + a_p z^p$, $b(z) = b_0 + b_1 z + \dots + b_q z^q$, $b_0 = 1$, $p \leq q$, a_l ($l = 0, 1, \dots, p$) and b_j ($j = 0, 1, \dots, q$) are real coefficients of two polynomials $a(z)$ and $b(z)$ of orders p and q , respectively. The approximation $G_{[p,q]}(z)$ of the IIR filter $G(z)$ implies that the input wavelet $\{w_l\}_{l=0}^{\infty}$ and the output signal $\{s_k\}_{k=0}^{\infty}$ must satisfy the recursion formula $\sum a_l w_{k-l} - \sum b_j s_{k-j} = s_k$, $k = 0, 1, 2, \dots$. A finite number $N > p + q$ must be chosen in order to reconstruct the full Padé coefficients a_l 's and b_j 's given the data $\{w_l\}_{l=0}^{\infty}$ and $\{s_k\}_{k=0}^{\infty}$. Some coefficients a_l 's and b_j 's in the middle terms of $a(z)$ and $b(z)$ can be assumed to be zero. For a fixed integer number m (e.g., $m = 6$), we suppose that $a_m = a_{m+1} = \dots = a_{p-(m-2)} = a_{p-(m-1)} = 0$, $b_{m+1} = b_{m+2} = \dots = b_{q-(m-2)} = b_{q-(m-1)} = 0$. The unknown coefficient vectors $c_1 = (a_0, a_1, \dots, a_{(m-1)}, a_{p-m}, \dots, a_p)^T$ and $c_2 = (b_1, b_2, \dots, b_m, b_{q-m}, \dots, b_q)^T$ satisfy: $AC = A_1 c_1 + A_2 c_2 = s$

where $C = (c_1, c_2)^T$, $s = (s_0, s_1, \dots, s_N)^T$, and the matrices A_1 and A_2 with entries in terms of data $\{w_l\}_{l=0}^N$ and $\{s_k\}_{k=0}^N$. The reconstruction problem for finding C is an inverse problem. It is ill-posed and requires regularization to develop a stable numerical algorithm. We introduce a penalization term in the Tikhonov regularization functional $T^\lambda(C, s)$, so that the reconstruction problem is formulated as the following constrained least squares minimization problem with the regularization parameter $\lambda > 0$ chosen properly:

$$\min_c T^\lambda(C, s) = \min_c \{ \|AC - s\|^2 + \lambda^2 \|C\|^2 \} \quad \text{s. t. } |u_k - r_0| < \delta_0, \quad |v_j - r_1| < \delta_1, \quad k = 1, 2, \dots, p, \quad j = 1, 2, \dots, q.$$

Here u_k and v_j in the constraints are zeros and poles of the reconstructed $[p, q]$ -Padé approximation $G_{[p,q]}(z)$ of $G(z)$, $r_0 = (-\alpha/\beta)^{(1/2d)}$, $r_1 = (-1/\eta)^{(1/2d)}$. After reconstruction of the real coefficient vector C , we can extend it to a full coefficients of the function $G_{[p,q]}(z)$ by inserting zero coefficients, this gives $[p, q]$ -Padé approximation $G_{[p,q]}(z)$ of $G(z)$. The reconstructed function $G_{[p,q]}(z)$ can be used to estimate the reflectivity parameters and to recover the impulse wavelet using inverse filtering.

Numerical Simulations

To simulate the synthetic data - impulse response signal $\{s_j\}_{j=0}^\infty$, a Ricker wavelet with the dominant frequency 25Hz was generated for the input signal, and the parameters a , b and c were chosen as $a = 4/45$, $b = 11/18$, $c = 9/10$, $d = 50$, leading to the polynomial coefficients a_0 , a_{100} , b_{100} represented by $\alpha = 0.7$, $\beta = -0.08$, $\eta = -0.9$, respectively. The sample rate of the Ricker wavelet is 0.003 seconds, the wavelet length is 3.0 seconds, so that the total number of data is $N = 1001$. The left Fig. 2 shows the reconstruction of the 26 reduced Padé coefficients for $m = 6$ compared with the true coefficients of $G(z)$ for the order of $p = q = 104$ chosen in the inversion algorithm. Here $m = 6$, so there were $4m + 2 = 26$ Padé coefficients to be determined for the vector C . The values of a_0 , a_{100} , b_{100} are reconstructed very accurately when there is no noise in the data. The valid recovered poles of $G_{[p,q]}(z)$ are illustrated in the right Fig. 2. Here $\delta_1 = \delta_2 = 0.06$, all poles and zeros of $G(z)$ lie on the unit circle with radius $r_1 = 1.0011$, $r_2 = 1.022$ in the complex z -plane, respectively. The true and computed output signals $\{s_j\}_{j=0}^\infty$ (the IIR filtered Ricker wavelet) using the recovered $[p, q]$ -Padé coefficients fit fairly well for data with 8% noise (left Fig. 3). The IIR filtered Ricker wavelet exhibits multiple reflections in the 1D synthetic seismogram. The recovered function $G_{[p,q]}(z)$ as an inverse filter was used to reconstruct the original input Ricker wavelet (right Fig. 3). The reconstruction of the input Ricker wavelet is almost identical, with no difference between the theoretical and reconstructed functions when there is no noise in the data. Even for input data with adding 8% noise, the original input Ricker wavelet in the time interval $[0, 1.5]$ seconds was recovered very well using the reconstructed function $G_{[p,q]}(z)$ as an inverse filter. However, there are some oscillation events that occurred with amplitudes changing rapidly after $t = 0.15$ seconds, which need to be further studied.

Conclusions

We developed a new numerical inversion method for reconstruction of the z-transform function of a time-dependent minimum phase signal using Padé approximation. The approach is based on rational $[p, q]$ -Padé approximation of the z-transform function in the complex plane. The problem is formulated as a constrained least squares minimization problem with regularization constraints provided by the minimum phase signal (all poles and zeros of z-transform function lie outside the unit circle). The method was tested using a

Ricker wavelet to generate a minimum phase signal (IIR filtered wavelet) as synthetic input data. The performed numerical experiments for reconstruction of the z-transform function and its use as an inverse filter show the effectiveness of the presented approach. The Padé approximation technique may provide a method to pull out the effective filter that produces the multiple reflections in seismic data processing.

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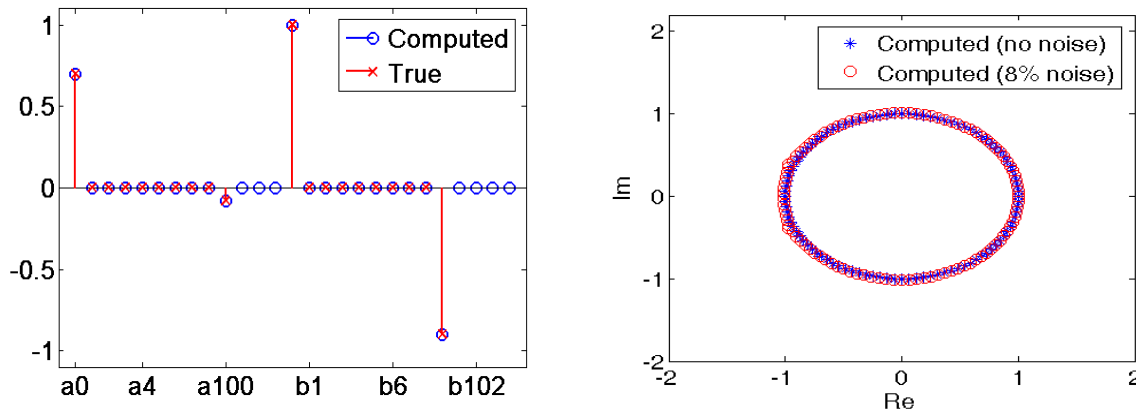


Figure 2. Reconstruction of Padé coefficients for data with no noise (left). Calculation of poles for function $G(z)$ (right).

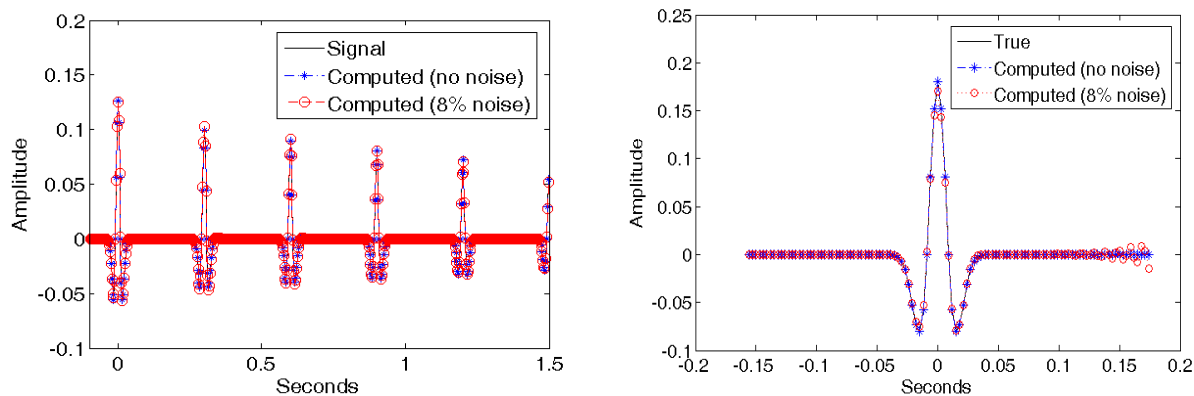


Figure 3. Reconstruction of the IIR filtered Ricker wavelet (left) and the Ricker wavelet with 25Hz dominant frequency (right).