Phase Unwrapping: A Review of Methods and a Novel Technique

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Computation of the complex cepstrum uses the logarithm, and in complex analysis the logarithm is multi-valued. To compensate, the phase of the complex variable that the logarithm operates on must be unwrapped. Phase unwrapping algorithms do exactly this, effectively turning the multi-valued logarithm into a well-behaved analytic function that is suitable for further manipulation. There are an abundance of phase unwrapping algorithms in the literature that either act on the phase itself, or on the complex, frequency domain signal from which the phase is derived. We introduce a further phase unwrapping algorithm that operates in the range of the z transform (often referred to as the w plane). In this domain, the branch-cut that is associated with wrapped phase is easily identified, and is used in our w plane phase unwrapping algorithm.

Introduction

In the 1970's and 1980's there was a great deal of interest in the complex cepstrum. The theory operates on amplitude and phase, thereby requiring them to behave analytically. While the amplitude is inherently analytic, the phase is not and must be unwrapped. More recently, Karam (2006) provided a thorough review of phase unwrapping methods, and introduced a hybrid technique using a combination of established algorithms. Meanwhile, Treitel et al. (2006) review the method of polynomial factorization (originally attributed to Steiglitz and Dickinson (1977,1982)) for the purpose of phase unwrapping. This renewed interest spurred our own investigations ultimately resulting in a novel technique.

We call the new technique w *plane phase unwrapping*. In many ways, it is simply a recasting of known solutions in a new domain, and unlike the root factoring method of Steiglitz and Dickinson (1977,1982), it does not (without appropriate use of interpolation) guarantee a correct solution. Moreover, it shares the spirit of algorithms published in McGowan and Kuc (1982) and Al-Nashi (1989) that use all information given by the z transform, rather than only its phase component. We begin with some motivation using the complex cepstrum. Then, we make a selective review of previous methods. Finally, we give a description of w plane phase unwrapping. During our exposition, we try to emphasize the potential problems (however obscure) that these algorithms must deal with, which in turn present potential reasons for failure.

Complex Cepstrum and Multi-Valued Functions

We concern ourselves with a transform that maps a signal from the time domain to the complex cepstrum where the signal is filtered before performing the inverse transform. This is a familiar strategy in signal and image processing. The difficulty lies in the fact that the transform, in part, uses the logarithm of a complex number. In complex analysis the logarithm is, due to the phase of a complex number, a multi-valued function. Here, we review the transform of the complex cepstrum, and analyze the mathematical domains (z plane and w plane) that are involved in the computations. It is in the w plane where the logarithm is applied, and where phase unwrapping is used to make it single-valued and, in turn, analytic.

The complex cepstrum of a discrete signal, $h(t_i)$, $i=1\ldots N$, is computed in the following steps: (1) compute the z transform of h, giving $w(z)$; (2) compute the logarithm of $w(z)$, giving $\log w(z)$; (3) compute the inverse z transform of $\log w(z)$ to find the complex cepstrum.

First, the z transform of h is

$$w(z) = h(t_0) + h(t_1)z + h(t_2)z^2 + \cdots + h(t_N)z^N \tag{1}$$

where we say that $z = x + iy$ is the z plane, and that $w(z) = u(z) + iv(z)$ is the w plane. We evaluate the polynomial in equation (1) for the unit circle in the z plane. That is, for $|z| = 1$ and $\arg(z) \in (-\pi, \pi]$. Second, we take the logarithm of the polynomial $w(z)$ so that

$$\log w(z) = Log|w(z)| + Arg(w(z)) + i2\pi n \quad , \quad n = 0, \pm 1, \pm 2, \ldots \quad . \tag{2}$$

Third, we compute the inverse z transform of equation (2), $\hat{h} = w^{-1}(\log(w(z)))$, where $\hat{h}$ is the complex cepstrum. For an application of the complex cepstrum in geophysics, please see Ulrych (1971).

From equation (2), it is evident that the logarithm is multi-valued. $\mathrm{Arg}(w(z))$ is the principal argument of $w(z)$, and is in practice computed using the arctangent function so that $Arg(w(z)) \in (-\pi, \pi]$. This choice of principal argument means that the branch-cut in the w plane is the non-positive real axis. It is this branch-cut that manifests as 2π discontinuities in phase, and various unwrapping methods remove its effect, allowing us to work with the logarithm as if it were analytic.

Selective Review of Phase Unwrapping Algorithms

Our introduction alludes to a variety of algorithms for phase unwrapping. Arguably, the simplest of these make use of the expected continuity in the analytic phase (Oppenheim and Schafer, 1989; Tribolet, 1977). More involved algorithms use the roots of the z transform polynomial (equation (1)); either the complex roots (Steiglitz and Dickenson, 1982), or the roots of the real component of $w(z)$ (McGowan and Kuc, 1982). Separately, Al-Nashi (1989) identifies and incorporates sharp zeros into his phase unwrapping algorithm.

The oft referenced algorithm of Tribolet (1977) uses the fundamental theorem of integral calculus to identify and remove discontinuities caused by the w plane branch-cut. This involves a numerical integration, and Tribolet (1977) ensures its validity through comparison, plus or minus integer multiples of 2π, to the principal argument . In essence, the Tribolet (1977) algorithm is similar to Oppenheim and Schafer (1989). Both use the phase of $w(z)$ to search for discontinuities created by the branch-cut in the w plane; but, the Tribolet (1977) method recognizes when $w(z)$ requires further interpolation.

Next, we consider the algorithms of McGowan and Kuc (1982) and Steiglitz and Dickinson (1982) that use the roots of $w(z)$ for phase unwrapping. In McGowan and Kuc, the roots of $u(z)$ (the real component of $w(z)$) are used. As usual, the wrapped phase is computed using the inverse tangent function, $\arg(w(z)) = a\tan(v(x,y)/u(x,y))$, and π is either added or subtracted as z passes through singularities of $v(x,y)/u(x,y)$. Thus the method must find the roots of $u(x,y)$. Conversely, Steiglitz and Dickenson (1982) factorize $w(z)$ so that each factor has analytic phase; their sum is the total phase, and is also analytic. That is, a sum of analytic functions is, itself, analytic. It is easy to argue that this represents the best possible algorithm for phase unwrapping in that it provides a guaranteed result. The downside is its need to factorize potentially large polynomials; from a practical point of view, not all time series need be factorizable. That said, there exist very fast algorithms for factoring one dimensional polynomials (e.g. Sitton et al., 2003).

Al-Nashi (1989) recognized that in addition to 2π discontinuities caused by the w plane branch-cut, there may also be π discontinuities generated by the presence of, so called, *sharp zeros* in the z plane. Sharp zeros are roots of $w(z)$ that fall close to the unit circle. The improbable zero that falls on the unit circle causes a π discontinuity. Algorithms that rely on the continuity of phase may mistakenly attribute the effects of sharp zeros to the w plane branch-cut; or alternatively, they may miss the effect of the branch-cut where $\mathrm{Arg}(w(z))$ is obscured by sharp zeros. To avoid this miss-classification, Al-Nashi (1989) designed an algorithm that incorporates the locations of sharp zeros.

The overview of methods presented here is not complete. Moreover, some of the details of each algorithm are left to the cited papers, and we encourage the interested reader to pursue these.

W Plane Phase Unwrapping

Here, we introduce a novel algorithm for phase unwrapping, called w plane phase unwrapping. It adds an algorithm to a problem with many existing solutions; none-the-less, we hope that it provides further insight, and is pleasing for its simplicity and straight forward extension to higher dimensions. The logic of the algorithm is short. When the contour of the filter in the w plane crosses the branch-cut, we adjust n in equation (2). Unlike the polynomial factorization algorithm of Steiglitz and Dickenson (1982), w plane phase unwrapping may require studious interpolation to guarantee a correct answer.

To proceed, we define $\omega = \arg(z)$, and use ω to parameterize the filter's w plane contour so that

$$w(z) = (u(z) + iv(z)) = w(\omega) = (u(\omega) + iv(\omega)) \quad , \quad z = e^{i\omega} \quad , \quad \omega \in (-\pi, \pi].$$

In practice, ω takes on a finite and ordered set of values, $\omega \in \{\omega_0, \omega_1, \ldots, \omega_N\}$ where $\omega_0 = -\pi + \varepsilon$, $\varepsilon > 0$ and $\omega_N = \pi$. The algorithm proceeds by comparing $w(\omega)$ for successive instances of ω. In particular, let $n_0 = 0$, and if (1) $u(\omega_{i-1}) \le 0$, $u(\omega_i) \le 0$ and $v(\omega_{i-1})v(\omega_i) < 0$, then let $n_i = n_{i-1} - \mathrm{sign}(v(\omega_{i-1}))$. The unwrapped phase is $\arg(w(\omega_i)) = Arg(w(\omega_i)) + i2\pi n_i$. If $v(\omega_{i-1})v(\omega_i) < 0$; but, if either (2) $u(\omega_{i-1}) < 0$ and $u(\omega_i) > 0$, or (3) $u(\omega_{i-1}) > 0$ and $u(\omega_i) < 0$ apply, then the result is ambiguous. In these two ambiguous cases, (2) and (3), it is not certain if the contour crosses the real axis $v = 0$ to the left or right of the imaginary axis $u = 0$. We note that the ambiguous case in (2) and (3) correspond to filters with sharp zeros in the z plane. In the cases of (2) or (3), then like Tribolet (1977), $w(z)$ must be interpolated to remove the ambiguity, and return us to the unambiguous condition (1).

In Figure 1, we show the application of the algorithm to a short filter where it is easy to illustrate each crossing of the branch cut. Figure 1a shows the roots of $w(z)$ in the z plane. Figure 1b shows the corresponding contour $w(z)$ in the w plane where we identify the crossings of the branch-cut. Figure 1c and 1d show the phase before and after application of w plane phase unwrapping.

We have implemented the straight forward extension of w plane phase unwrapping to 2D. We omit details and examples for the sake of brevity.

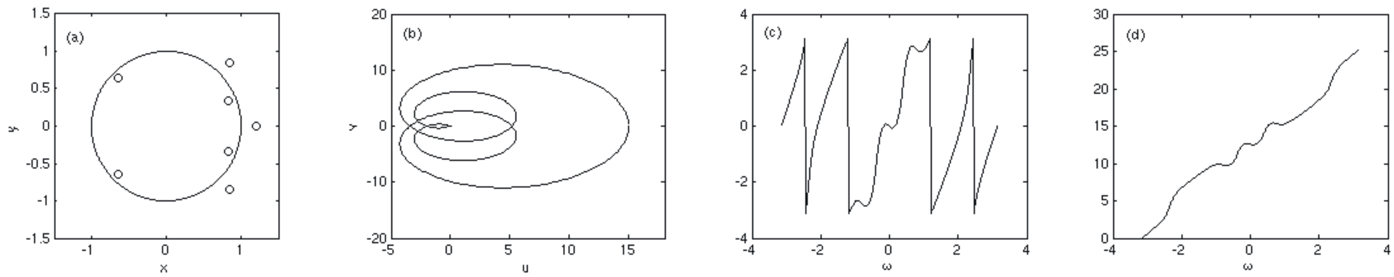

Figure 1. Example of w plane phase unwrapping. In (a), we plot the roots of the filter in the z plane. (b) The same filter in the w plane. (c) The wrapped phase, and (d) the unwrapped phase.

Discussion

We introduce w plane phase unwrapping. While we do not presume to recommend its use in place of existing algorithms, we believe it to be a novel approach. Our algorithm operates in the w plane. In this domain, the 2π jumps that phase unwrapping algorithms are tasked with removing are naturally identified with the branch-cut. Indeed, all phase unwrapping algorithms must account for this branch-cut, and here we do so in what we consider to be the most appropriate domain.

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