Implicit Preconditioning for Least-squares Nonstationary Phase Shift

Marcus R. Wilson*
University of Calgary, Calgary, Alberta, Canada
wilsonmr@ucalgary.ca
and
Robert J. Ferguson
University of Calgary, Calgary, Alberta, Canada

Summary

We present an iterative inversion to correct for surface statics and irregular trace spacing. The algorithm improves on previous inversions by deriving an implicit preconditioner for the least-squares operator, which reduces the required number of conjugate gradient iterations at the cost of introducing an approximation to the evanescent filter. We observe a substantial reduction in the number of iterations required in the low frequency range, where the original operator is poorly conditioned, but artifacts are observed in the new scheme as a result of the approximation. This speedup is more significant for larger surveys, as the number of conjugate gradient iterations required increases very slowly as we add traces. Decimated traces are reconstructed even when no smoother is applied, with larger depth steps resulting in better regularization.

Introduction

To correct for surface statics and irregular trace spacing before migration, Ferguson (2006) presents an inversion algorithm based on the phase shift method of Gazdag (1978). Acquired seismic data is extrapolated recursively through the near surface by weighted-damped least squares. The result is a regularly sampled wavefield at a flat datum, which can then be imaged using migration techniques that use the fast Fourier transform.

Implementation of this operator as a matrix is prohibitively costly to compute. The associated Hessian is constructed at the cost of matrix-matrix multiplication, with complexity $O(n^3)$, where n is the number of traces. Inversion of the Hessian by Gaussian elimination also has complexity $O(n^3)$ (Wilson and Ferguson, 2010). These computations have to be repeated for every depth step and every frequency. We can reduce these costs by recasting the problem in a conjugate gradient framework, replacing matrices with function calls, where the Hessian is applied as a forward operator, and the extrapolated wavefield can be computed by an iterative search. The cost of the resulting inversion scheme is the cost of applying the forward operator times the number of iterations required for an acceptable approximation.

Wilson and Ferguson (2010) presents an application of this inversion scheme. The cost of applying the forward operator can be reduced to $O(vn\log n)$, where v is the number of reference velocities in the velocity model. The algorithm converges in a reasonable number of iterations for large frequencies, but fails to converge quickly for lower frequencies. Wilson and Ferguson (2010) postulate that the poor convergence in the lower frequencies is caused by the evanescent filter embedded in the phase shift operator. Here we will derive a preconditioning scheme by which the effects of this filter can be mitigated, and we observe the effects of this change on the convergence rate of the conjugate gradient method.

Non-stationary Phase Shift Operators

In a layered medium, the phase shift operator acts within a layer on a monochromatic wavefield φ_z at depth z by way of a spatial fast Fourier transform, followed by multiplication by an extrapolation matrix, then an inverse fast Fourier transform. Written as matrices, we have

$$P_{\Delta z}(\varphi_z) = [IFT][\alpha_{\Delta z}][FT]\varphi_z \,. \tag{1}$$

Here $[\alpha_{\Delta z}]$ is a diagonal matrix that applies the phase shift operator in the wavelike region, where $|\omega/v_z| \geq |k_x|$, and attenuates energy in the evanescent region, where $|\omega/v_z| < |k_x|$. The diagonal elements of $[\alpha_{\Delta z}]$ are computed from the layer velocity v_z and the input wavenumber k_x using the formula,

$$\alpha_{\Delta z}(k_x, v_z) = \exp(i\Delta z k_z)\,, \tag{2}$$

where the wavenumber k_z is given by,

$$k_z = \mathrm{Re}\left\{\sqrt{\left(\frac{\omega}{v_z}\right)^2 - k_z^{\,2}}\right\} + i\,\mathrm{sgn}(\Delta z)\,\mathrm{Im}\left\{\sqrt{\left(\frac{\omega}{v_z}\right)^2 - k_z^{\,2}}\right\}. \tag{3}$$

To accommodate lateral velocity variation, we use a set of constant velocity windows, defined for a given reference velocity v by,

$$\Omega_v(x) = \begin{cases} 1 & \text{if} \quad v(x) = v \\ 0 & \text{if} \quad v(x) \neq v \end{cases}, \tag{4}$$

and the phase shift operator becomes (Margrave and Ferguson, 1999),

$$P_{\Delta z}(\varphi_z) = \sum_v [\Omega_v][IFT][\alpha_{\Delta z}]_v[FT]\varphi_z \,. \tag{5}$$

Evanescent Filter

The wavefield extrapolator $\alpha_{\Delta z}$ can be factored into two parts: a complex exponential that performs the phase shift, and a negative real exponential, which acts as an evanescent filter. Wilson and Ferguson (2010) notes that convergence of the least squares inversion of $P_{\Delta z}$ was fast for high frequencies, and slow for low frequencies, where the data vector crosses into the evanescent region. It was postulated that this slow convergence was the result of the evanescent filter. To overcome this difficulty, we can factor the filter out of the least-squares Hessian. To that end, express $\alpha_{\Delta z}$ as,

$$\alpha_{\Delta z} = \exp(i\Delta z\,\mathrm{Re}\{k_z\})\exp(-i|\Delta z|\,\mathrm{Im}\{k_z\})\,. \tag{6}$$

So we can factor $[\alpha_{\Delta z}]$ into two diagonal matrices: $[\alpha_{\Delta z}^{P}]$, which applies the phase shift, and $[\alpha_{\Delta z}^{F}]$, which applies the filter.

$$[\alpha_{\Delta z}] = [\alpha_{\Delta z}^{P}][\alpha_{\Delta z}^{F}] \tag{7}$$

Now if we set each $[\alpha_{\Delta z}^{F}]$ to filter with respect to the highest reference velocity, we can factor the matrix $P_{\Delta z}$,

$$P_{\Delta z}(\varphi_z) = \{\sum_v [\Omega_v][IFT][\alpha_{\Delta z}^{P}]_v\}\{[\alpha_{\Delta z}^{F}][FT]\varphi_z\} = \{Q_{\Delta z}\}\{F\varphi_z\}\,. \tag{8}$$

Least Squares Minimization by Conjugate Gradients

If the seismic wavefield is regularly sampled, the operator $P_{\Delta z}$ can be applied to quickly extrapolate the data downward into the subsurface. For an irregularly sampled wavefield, Ferguson (2006) solves for the extrapolated wavefield by weighted damped least-squares. For a survey with n traces, this requires us to solve an nxn matrix for every depth step and every frequency. Solving this matrix by Gaussian elimination is prohibitively costly when n is large. We can reduce this cost somewhat by using conjugate gradients to solve the system, using an upward phase shift $P_{-\Delta z}$ as the forward operator. The resulting operator is poorly conditioned in the low frequencies (Wilson and Ferguson, 2010), so it requires a large number of conjugate gradient iterations to converge to an acceptable solution. The transformed phase shift operator $Q_{\Delta z}$ defined in Equation 8 is much better conditioned, so we can solve the linear system for $F\varphi_z$ in fewer iterations. The extrapolated wavefield can then be solved by applying the inverse of F, which is easy to compute.

Example

To demonstrate the effect of preconditioning on our linear system (Equation 5), we generate an arbitrary source wavefield of $n = 256$ traces (Figure 1), and Fourier transform in time. The operator $P_{\Delta z}$ in Equation 5 is applied to the resulting monochromatic wavefields, and a random selection of 30% of the traces is set to zero to model irregular trace spacing. The resulting synthetic data is given in Figure 2. We attempt to recover the source wavefield in two ways. First, we apply the conjugate gradient method to solve the Hessian matrix generated by $P_{\Delta z}$ (Wilson and Ferguson, 2010). The difference between the source and recovered wavefields is shown in Figure 3. Next, we solve for $F\varphi_z$ by applying conjugate gradients to the Hessian matrix generated by $Q_{\Delta z}$ as defined in Equation 8, and the recovered wavefield is computed by applying the inverse of F. The error of this scheme is shown in Figure 4.

In Figure 3, migration artifacts can be observed in the error plot, localized around any large gaps in trace coverage, but errors are generally very small. In Figure 4, the error from the implicit scheme is more coherent, and shows a difference in the event amplitudes between the source and recovered wavefields. This may be caused by the approximation to the evanescent filter we had to make in order to factor $P_{\Delta z}$, or by rounding errors that result from multiplying by the poorly conditioned inverse of the filter F.

The number of iterations required at each frequency is shown in Figure 5 for both methods. Note that the preconditioned scheme converges for all frequencies, and gives a result in fewer iterations than the original scheme. In Figure 6, we see how the number of iterations required for the implicit scheme increases with the number of traces. The blue best fit curve shows that the number of iterations grows slowly with the size of the problem, which is preferred if we wish to construct an inversion scheme that is fast for large surveys.

Conclusions

We have implemented an implicit preconditioned conjugate gradient scheme that makes least-squares nonstationary phase shift run very fast on large trace gathers. The implicit scheme is much faster than the standard scheme, but requires us to make an approximation that causes artifacts in the resulting output wavefield. The preconditioning operator handles the problem of poor convergence in the evanescent region by factoring an approximation of the evanescent filter out of the least-squares Hessian. This preconditioned conjugate gradient scheme indicates that fast nonstationary wavefield propagation by least squares is possible, although this particular scheme does not give the most accurate result. We postulate that preconditioned schemes exist which give a comparable speedup without sacrificing accuracy, although these schemes have yet to be determined.

Acknowledgements

The authors wish to thank the sponsors, faculty, staff and students of the Consortium for Research in Elastic Wave Exploration Seismology (CREWES), and the Natural Sciences and Engineering Research Council of Canada (NSERC, CRDPJ 379744-08) for their support of this work.

References

Ferguson, R. J., 2006, Regularization and datuming of seismic data by weighted, damped least squares: Geophysics, **71**, U67-U76.

Gazdag, J., 1978, Wave equation migration with the phase-shift method: Geophysics, **43**, 1342-1351.

Margrave, G.F., and Ferguson, R. J., 1999, Prestack depth migration by symmetric nonstationary phase shift: CREWES Research Report, No. 11.

Wilson, M. R., and Ferguson, R. J., 2010, Trace interpolation and elevation statics by conjugate gradients: GeoCanada 2010 Conference Abstracts.

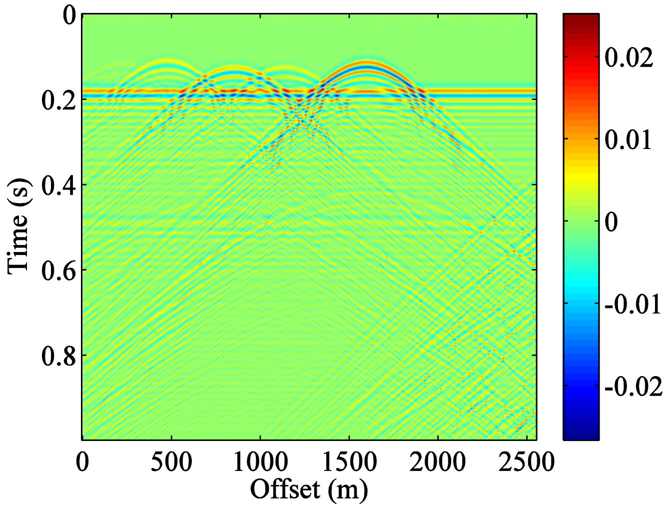

Figure 1: Source Wavefield

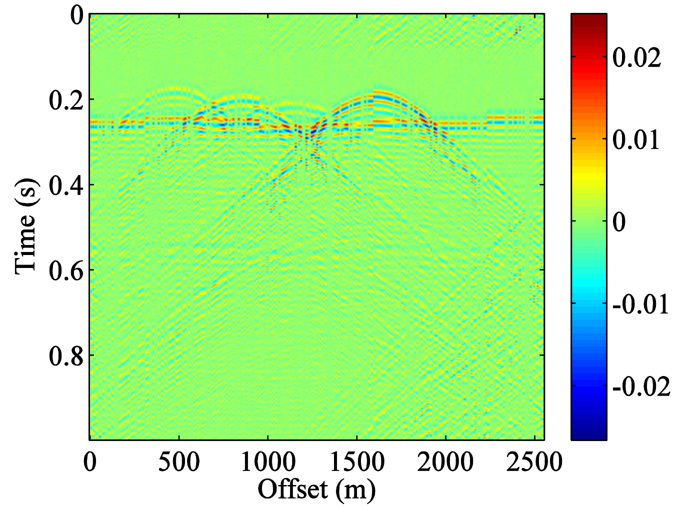

Figure 2: Phase Shifted Data

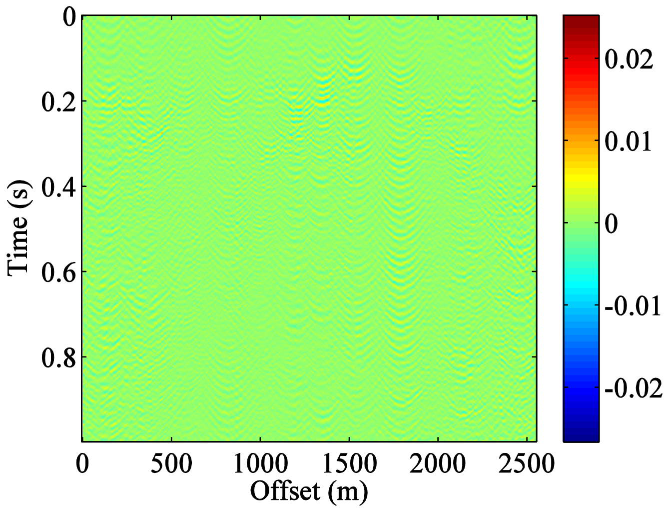

Figure 3: Recovered Wavefield Error – Standard Scheme

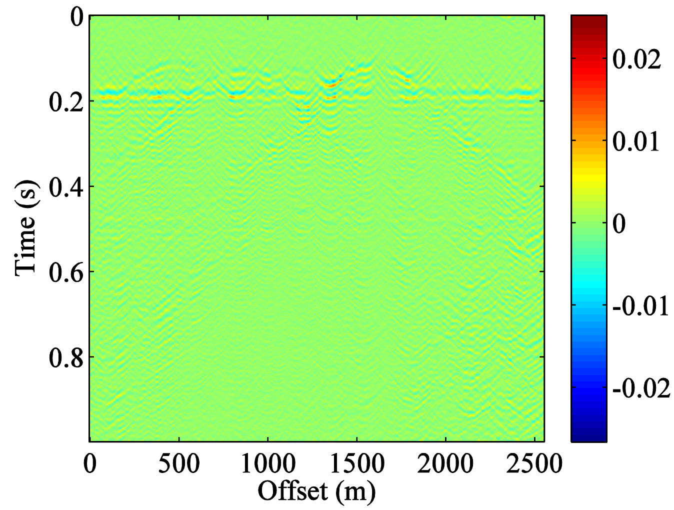

Figure 4: Recovered Wavefield Error – Implicit Scheme

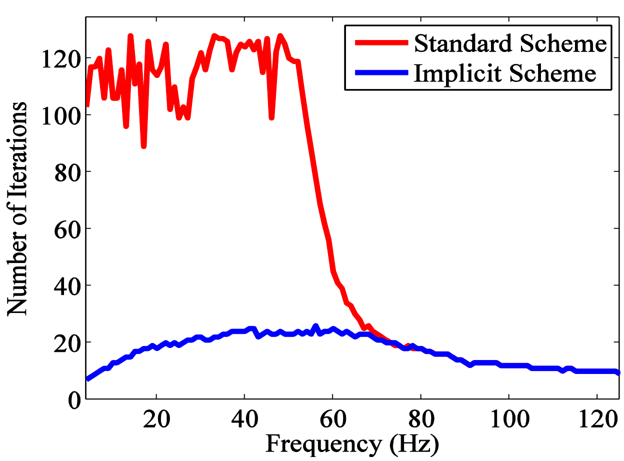

Figure 5: The number of iterations required per frequency for the standard and implicit schemes. The number of iterations required does not vary greatly with frequency for the implicit scheme.

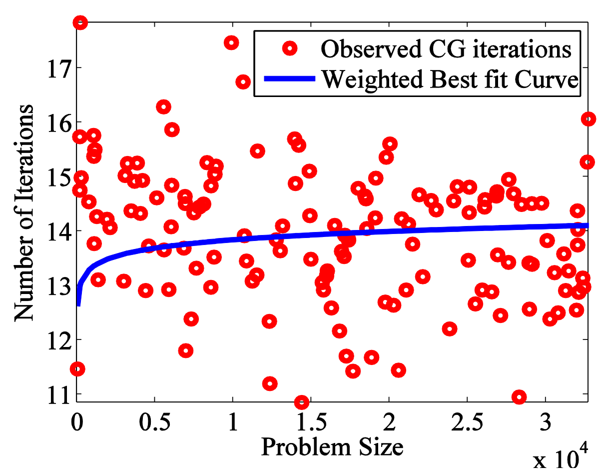

Figure 6: The number of iterations required for varying numbers of traces using the implicit scheme. The blue best fit curve estimates the number of iterations required per frequency for n traces is given by $C(n) = 11.96n^{0.0158}$.