

# Kirchhoff imaging in a matrix form. Part II: least squares migration

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## Summary

A visual method for construction of migration operator in a matrix form is introduced. Once the matrix is constructed, migration can be repeated just by a matrix-vector multiplication. This is useful in some applications that need several migration performances on data. Least squares migration is one of these applications. The shape of least squares migration matrix,  $G^T G$ , is presented as well.

## Introduction

The idea of using Kirchhoff modelling operator in the explicit forms is explained in part I in this report. Same methodology is implemented here to reconstruct Kirchhoff migration matrix. Considering each subsurface point as a diffraction point, Kirchhoff time migration collapses all diffracted energies to the scatterpoints using Double Squares Root (DSR) equation. The result is a properly migrated image. Since Kirchhoff modelling and migration are linear operators, they can be expressed in a matrix-vector multiplication scheme.

By defining the earth reflectivity model by  $m$ , and the Kirchhoff modelling operator by  $G$ , the seismic data,  $d$ , is given by,

$$d = Gm. \quad (1)$$

The inverse of this calculation should return the reflectivity back,

$$m = G^{-1}d. \quad (2)$$

However, calculation of inverse is not always a possible or easy task. First of all,  $G$  may not be a square matrix and invertible. Second problem in calculating  $G^{-1}$  is the large size of  $G$  matrix. The transpose (adjoint) of  $G$  matrix is  $G^T$ , and is easier to calculate than the exact inverse. The transpose is less sensitive to the imperfection of data; therefore, it does a better job than the inverse (Claerbout, 2005). Migration methods are essentially transpose process. In the case of Kirchhoff modelling, the transpose process is prestack Kirchhoff migration which is defined by,

$$\hat{m} = G^T d, \quad (3)$$

where  $\hat{m}$  is the migrated image and  $G^T$  is the migration operator.

In least squares migration,  $m$  can be estimated by minimizing the difference between the observed data  $d$ , and the modeled data  $G\hat{m}$ , expressed by  $|G\hat{m} - d|$ , where  $\hat{m}$  is an approximation to  $m$ . Finding  $m$  is feasible by solving the following normal equation:

$$G^T Gm - G^T d = 0. \quad (4)$$

Solving equation 4 gives the least squares minimum norm solution,  $m_{LS}$ , to the problem:

$$m_{LS} = (G^T G)^{-1} G^T d. \quad (5)$$

## Construction of $G^T$ matrix

The transpose of modelling matrix is the migration matrix,  $G^T$ . However, the construction of migration

matrix is easier to understand. For more information readers are being referred to Yousefzadeh and Bancroft (2010, 2011).

For simplicity assume one shot gather with one hyperbola in eight traces, each with six samples as:

$$\mathbf{d} = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (6)$$

In order to do Kirchhoff migration, all samples on this hyperbola (2s) must be added and the result ( $8 \times 2 = 16$ ) placed on the apex of the hyperbola, in the position of row # 1 and column # 5 of the image domain. Lets  $\tilde{\mathbf{G}}_{1,5}^T$  be a transposed version of  $\mathbf{d}$ , but with ones instead of the non-zero elements as seen in equation 7:

$$\tilde{\mathbf{G}}_{1,5}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \quad (7)$$

Index 1,5 in  $\tilde{\mathbf{G}}_{1,5}^T$  means that we are looking for a migration operator for the scatterpoint at row # 1 and column # 5 of our model. Then, multiplication of  $\tilde{\mathbf{G}}_{1,5}^T$  by  $\mathbf{d}$  gives the following matrix:

$$\tilde{\mathbf{G}}_{1,5}^T \mathbf{d} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \quad (8)$$

Summation of the diagonal elements of  $\tilde{\mathbf{G}}_{1,5}^T \mathbf{d}$  matrix is equal to the number that must go to the row # 1 and column # 5 in the migration image:

$$\hat{\mathbf{m}}_{1,5} = \Sigma[\text{diag}(\tilde{\mathbf{G}}_{1,5}^T \mathbf{d})] = [16]. \quad (9)$$

This procedure is valid only for one hyperbola in data and one point in the image domain,  $\hat{\mathbf{m}}_{1,5}$ . Practically we need a matrix to multiply to the whole data to get the desired migration image:

$$\hat{\mathbf{m}}_{1,5} = \mathbf{G}_{1,5}^T \mathbf{d}, \quad (10)$$

where  $\hat{\mathbf{m}}_{1,5}$  is the migration image at row #1 and column # 5. Data is a matrix with  $M_{trc}$  traces, each trace in a column, and  $N_t$  samples per column. To do matrix multiplication in equation 10,  $\mathbf{d}$  must be vectorized to  $\mathbf{d}_v$ . To have the data as a vector, we put column  $i + 1$  of  $\mathbf{d}$  below column  $i$  for all columns. Consequently,  $\tilde{\mathbf{G}}_{1,5}^T$  must also be a row vector. If we put all rows of  $\tilde{\mathbf{G}}_{1,5}^T$  next to each other, the desired vector is achieved:

$$\tilde{\mathbf{G}}_{v1,5}^T = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ \dots \ 1 \ 0 \ 0]. \quad (11)$$

Now multiplication of  $\tilde{\mathbf{G}}_{v1,5}^T$  by  $\mathbf{d}_v$ , is equal to the summation along the diagonal elements of  $\tilde{\mathbf{G}}_{1,5}^T \mathbf{d}$  matrix in equation 8. Therefore, calculating the diagonal elements of  $\tilde{\mathbf{G}}_{1,5}^T \mathbf{d}$  matrix, can be replaced by the multiplication of  $\tilde{\mathbf{G}}_{v1,5}^T$  row vector to the  $\mathbf{d}_v$  column vector. This is equivalent to the summation of amplitudes along the flanks of the diffraction hyperbola:

$$\hat{\mathbf{m}}_{1,5} = \tilde{\mathbf{G}}_{v1,5}^T \mathbf{d}_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & \dots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ \vdots \\ 2 \\ 0 \\ 0 \end{bmatrix} = [16]. \quad (12)$$

Vector  $\mathbf{d}_v$  has  $N_t \times M_{trc}$  elements.  $\tilde{\mathbf{G}}_{v1,5}^T$  is a vector with the same length. The resulted  $\hat{\mathbf{m}}_{1,5}$  is a number corresponding to one point in the image domain, at the hyperbola apex. In reality, there are  $N_x \times N_z$  possibilities for the position of hyperbola apexes. Therefore, if  $\tilde{\mathbf{G}}_{vi,j}^T$  row vector is for migration of one point in the image domain at row #  $i$  and column #  $j$ , then for each image point we have to have another row vector for  $\tilde{\mathbf{G}}^T$ . This can be done by adding additional rows to  $\mathbf{G}^T$ . If each row in the  $\mathbf{G}^T$  matrix represents a point in the migrated image then  $\mathbf{G}^T$  will include  $N_x \times N_z$  rows. The result of multiplication of  $\mathbf{G}^T$ , a  $N_x \times N_z$  by  $N_t \times M_{trc}$  matrix, to the  $\mathbf{d}_v$ , a  $N_t \times M_{trc}$  vector is a  $N_x \times N_z$  vector:

$$\mathbf{G}^T \mathbf{d}_v = \hat{\mathbf{m}}_v \Rightarrow \begin{matrix} N_x \\ N_z \end{matrix} \times \underbrace{\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}}_{N_t \times M_{trc}} \times \begin{matrix} N_t \\ M_{trc} \end{matrix} \left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \right\} = \begin{matrix} N_x \\ N_z \end{matrix} \times \left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} \right\}. \quad (13)$$

This vector must be reshaped as a  $N_x$  by  $N_z$  matrix to form the migrated image on an interpretable section. In vectorizing  $\mathbf{d}_v$ , we put the seismic traces below each other by putting the second trace below the first one and the third trace below the second one and so on. If the same vectorization for the image,  $\hat{\mathbf{m}}_v$ , is desired, it must be considered in the construction of  $\mathbf{G}^T$  matrix in advance. Each row in  $\mathbf{G}^T$  matrix represents one sample in the image vector  $\hat{\mathbf{m}}_v$ . By putting all rows corresponded to first column of the image matrix below each other and then repeating this procedure for the next column of the image matrix,  $\hat{\mathbf{m}}_v$  is vectorized in the same manner as  $\mathbf{d}_v$  is vectorized.

In calculation of  $\mathbf{G}^T$ , migration weight and other migration parameters must be included. Weight can be added to  $\mathbf{G}^T$  matrix by replacement of ones with the proper amplitude scales. Migration weights are applied to the data in the most internal loop of migration algorithm. Consequently,  $\mathbf{G}^T$  can be calculated only by going to the most internal loop and replacing the  $\mathbf{G}^T$  elements by the calculated migration weight. If instead of using the DSR equation, we use a ray tracing program, this technique can be extended to the prestack Kirchhoff depth migration.

### Example of $\mathbf{G}^T$ , and $\mathbf{G}^T \mathbf{G}$ matrices

As an example, suppose the assumed geometry includes only one source, 15 receivers, 15 CMPs, 200 pseudo depth samples and our data has 300 samples per trace. This geometry is shown in Figure 1. Migration is a  $4500 \times 3000$  matrix, a large matrix for such a small geometry. This matrix is shown in Figure 2. Figures 3a and 3b show the inversion matrix with the geometry in Figure 1.



Figure 1: A 2D model including one source (red triangle) on the top of a scatterpoint (blue circle) and eight receivers (flags) on the earth surface.

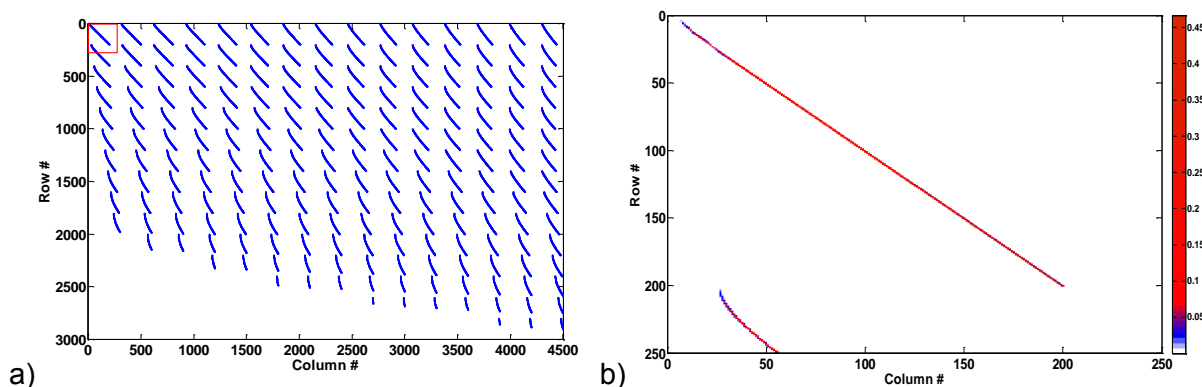


Figure 2: a) Non-zero elements of matrix  $G^T$ , b) Close up of the first 250 rows and columns.

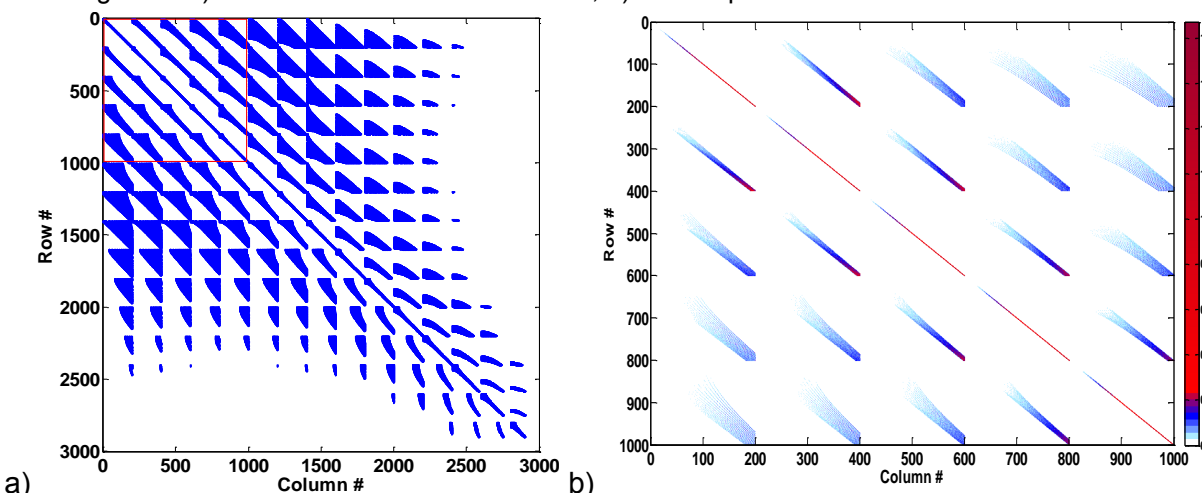


Figure 3: a) Non-zero elements of matrix  $G^T G$ , b) Close up of the first 1000 rows and columns.

## Conclusions

A technique for creating Kirchhoff migration and inversion matrices is shown without the requirement of going into the mathematics of Kirchhoff migration or Green functions. Once it is calculated, migration can be easily repeated just by multiplication of the matrix to data vector. However, the migration or modelling matrix is too big or dense which make it impractical for many real datasets.

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