Estimation of *Q*-factor and phase velocity using the recovered stress-strain relaxation spectrum

Dali Zhang*
Department of Mathematics and Statistics, University of Calgary
dlzhang@math.ucalgary.ca

Michael P. Lamoureux
Department of Mathematics and Statistics, University of Calgary
mikel@math.ucalgary.ca

and

Gary F. Margrave
Department of Geoscience, University of Calgary
margrave@ucalgary.ca

Summary

The paper presents a numerical inversion method for estimation of *Q*-factor and phase velocity in viscoelastic media using recovery of relaxation spectrum from measured or computed complex velocity or complex modulus of the medium. Mathematically the problem is formulated as an inverse problem for reconstruction of spectral measure in the Stieltjes representation of the complex modulus using rational ($[p,q]$-Padé) approximation. The approximation is obtained by solving a constrained least squares minimization problem with regularization. The recovered stress-strain relaxation spectrum is applied to numerical calculation of frequency dependent *Q*-factor and frequency dependent phase velocity for a standard linear viscoelastic solid (Zener) model as well as a nearly constant-*Q* model which has a continuous spectrum. Numerical results show good agreement between theoretical and predicted values and demonstrate the validity of the algorithm. The method can be used for evaluating relaxation mechanisms in seismic wavefield simulation of viscoelastic media. The constructed lower order Padé approximation can be used for determination of the internal memory variables in time-domain finite difference (TDFD) numerical simulation of viscoelastic wave propagation.

Introduction

The inversion method of Padé approximation is based on a constrained least squares minimization algorithm, regularized by the constraints derived from the analytic Stieltjes integral representation of the complex modulus. Solution of the constrained minimization problem provides coefficients of a rational approximation to the spectral measure of the medium. This rational approximation is transformed into Padé approximation by partial fraction decomposition. The method can use as data the values of measured, or simulated complex modulus or complex velocity in certain interval of frequencies. The recovered lower order $[p,q]$-Padé approximation can be used for determination of the internal memory variables in TDFD numerical simulation of viscoelastic wave propagation. The developed technique together with finite difference modeling may eventually lead to a new simultaneous inversion technique for estimation of the frequency dependent complex velocities, *Q*-factors and phase velocities in anelastic attenuating media from vertical seismic profile (VSP) data in geophysics prospecting.

Q-factor modeling and rational approximation for inversion

We consider a plane compressional wave propagating in a homogeneous isotropic viscoelastic medium with constant material properties. The equation of motion and the relation between stress σ and strain ε for one-dimensional (1D) linear viscoelastic media are represented by

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \quad \sigma = \mathrm{M} * d\varepsilon = \int_{-\infty}^{t} \mathrm{M}(t-\tau) d\varepsilon(\tau), \quad \varepsilon = \frac{\partial u}{\partial x}$$

where ρ is the mass density, $u(x,t)$ is the displacement, $\mathrm{M}(t)$ is the relaxation function of the medium. In the frequency domain, the relation between stress σ and strain ε in a linear viscoelastic medium and the Q-factor can be formulated as

$$\sigma(\omega) = M(\omega)\varepsilon(\omega), \quad Q(\omega) = \mathrm{Re}\, M(\omega) / \mathrm{Im}\, M(\omega) = 1/\tan(\theta(\omega))$$

where $M(\omega)$ is the complex viscoelastic modulus, θ is the phase of M. The complex velocity $V(\omega)$ and the phase velocity $c(\omega)$ in an attenuating medium are given by (Carcione, 2007):

$$V(\omega) = \sqrt{M(\omega)/\rho}, \quad 1/c(\omega) = \mathrm{Re}\sqrt{\rho / M(\omega)}$$

respectively. $M(\omega)$ is uniquely determined by a given $Q(\omega)$ in a causal medium since $\mathrm{Re}\, M$ and $\mathrm{Im}\, M$ must obey a Kramers-Kronig relation. In seismic applications, $Q(\omega)$ is normally assumed to be frequency-independent or only slowly varying with frequency. The information about the relaxation spectrum of the medium is contained in the spectral measure $\eta(x)$ in the analytic Stieltjes integral representation of the complex modulus (Day & Minster, 1984):

$$G(s) = \frac{M_U - M(s/i)}{\delta M} = \int_0^\infty \frac{d\eta(x)}{s+x}, \text{ with the sum rule} \int_0^\infty \frac{d\eta(x)}{x} = 1, \; s = i\omega .$$

Here $i = \sqrt{-1}$, M_U is the unrelaxed modulus and δM is the relaxation of the modulus. The function $G(s)$ is analytic outside the negative real semi axis in the complex *s*-plane, all its singular points are in the interval $(-\infty, 0)$. The function $\eta(x)$ can be approximated by a step function with a finite number of steps, so that

$$d\eta(x) \cong d\hat{\eta}(x) = \sum_{n=1}^{q} A_n \delta(x + s_n) dx, \quad x \in (0, \infty) .$$

Thus, the approximation $\hat{G}(s)$ of the function $G(s)$ is given by

$$G(s) \cong \hat{G}(s) = \sum_{n=1}^{q} \frac{A_n}{s - s_n}, \quad \text{s. t.} \; -\infty < s_n < 0, \; 0 < \frac{A_n}{|s_n|} < 1, \; \sum \frac{A_n}{|s_n|} = 1 .$$

Here s_n is the *n-th* simple pole on the negative real semi axis with positive residue A_n, *q* is the total number of poles. The approximation of the complex modulus $M(\omega)$ is obtained as

$$M(\omega) \cong M_U - \delta M \sum_{n=1}^{q} \frac{A_n}{i\omega - s_n} .$$

Therefore, the Q-factor and $V(\omega)$ can be estimated in terms of A_n and s_n as

$$Q(\omega) \cong \frac{\mathrm{Re}\{M_U - \delta M \sum A_n/(i\omega - s_n)\}}{\mathrm{Im}\{M_U - \delta M \sum A_n/(i\omega - s_n)\}}, \; V(\omega) \cong V^c(\omega) = \frac{1}{\sqrt{\rho}} \sqrt{M_U - \delta M \sum_{n=1}^{q} \frac{A_n}{i\omega - s_n}} .$$

Then the phase velocity can be estimated as $c(\omega) \cong 1/\mathrm{Re} V^c(\omega)$. A new numerical inversion algorithm for reconstruction of the measure $\eta(x)$ is developed using constrained $[p,q]$-Padé approximation (Baker, 1996) of $\hat{G}(s)$ and its partial fraction decomposition. The approximation has the form of $G(s) \cong \hat{G}(s) = a_p(s) / b_q(s)$, $p < q$; p, q are the orders of real polynomials $a_p(s)$ and $b_q(s)$, respectively. We assume that the complex velocity $V(\omega)$ or complex modulus $M(\omega)$ can be measured at sample data point of frequencies ω_k $(k = 1,2,\cdots,N)$ where N is the total number of data points. Measurements $V(\omega_k)$ or $M(\omega_k)$ can be transformed to $G(z_k)$ in the complex s-plane, $z_k = i\omega_k$, corresponding to each sample frequency. Thus we have data pairs (z_k, d_k), $d_k = G(z_k)$. The unknown coefficients of $a_p(s)$ and $b_q(s)$ are determined by solving the linear system of equations: $Sc = g$. Here the vector c contains all normalized coefficients of $a_p(s)$ and $b_q(s)$, $g = g_R + ig_I$ and $S = S_R + iS_I$, subindices R and I indicating the real and imaginary parts of the matrices with entries in terms of data. The reconstruction problem of determining the coefficient vector c is an inverse problem; it is ill-posed. To derive a stable numerical inversion algorithm, a penalization term was introduced in the Tikhonov regularization functional (Tikhonov et al., 1977) and the reconstruction problem is formulated as the constrained least squares minimization problem with the regularization parameter $\lambda > 0$ chosen properly (Zhang & Cherkaev, 2009; Zhang & Lamoureux et al., 2009):

$$\min_c \{\| S_R c - g_R \|^2 + \| S_I c - g_I \|^2 + \lambda^2 \| c \|^2\} \text{ s. t. } -\infty < s_n < 0,\ 0 < \frac{A_n}{|s_n|} < 1,\ n = 1,2,\cdots,q.$$

After recovery of the coefficient vector *c* of the rational function approximation $\hat{G}(s)$, its decomposition into partial fractions, gives $[p,q]$-Padé approximation of $G(s)$. Then the *Q*-factor and phase velocity can be calculated using the above derived formulas.

Results

In the first example we consider inverse modeling of *Q*-factor for a standard linear solid (SLS) (Generalized Zener) model. The values of material strain-stress relaxation times were chosen from (Tal-Ezer et al., 1990) to calculate the synthetic complex modulus $M(\omega)$ with five relaxation mechanisms to yield a constant $Q = 100$ at 50 data points (frequency range: 2~50Hz), ρ =2000 kg/m^3 and $M_R = M_U - \delta M = 8$ Gpa. The function $G(s)$ in this analytic model has a five-term partial fractions form. A_n and s_n are reconstructed almost exactly in the case of $q \geq 5$ when there is no noise in the data. The recovered poles and residues are further used to covert the values of the strain-stress relaxation times with the number of relaxation mechanisms being less than five using the low order $[p,q]$-Padé approximant method for evaluating *Q*-factor and phase velocity. Fig. 1 illustrates the numerical results for sensitivity analysis of the estimated *Q*-factors and phase velocities using data with added noise. The results agree with the published simulations in (Tal-Ezer et al., 1990). To further examine the effectiveness of the developed inversion method we consider a nearly constant $Q = 21$ model with a continuous relaxation spectrum. The complex velocity measurements were simulated at 50 data points (frequency range: 0.01~100Hz). Fig. 2 shows the estimation of *Q*-factors and phase velocities using Padé approximation for $q = 4$ and $q = 5$.

Conclusions

A new numerical inversion method for estimation of *Q*-factor and phase velocity in homogeneous dissipating media was developed using Padé approximation. The inverse problem was solved as a constrained least squares minimization problem with regularization constraints provided by the Stieltjes integral representation of the complex modulus. The method was tested using analytical models of viscoelastic media with a continuous spectrum as well as a standard linear solid (Zener) model. The numerical results demonstrate the effectiveness of the developed approach. The method can be used for identification of relaxation parameters of viscoelastic materials from measurements of complex velocity or complex modulus. The recovered relaxation mechanisms can be used for numerical modeling of seismic wavefields in viscoelastic media.

Acknowledgements

We gratefully acknowledge generous support from NSERC, MITACS, PIMS, and sponsors of the POTSI and CREWES projects. The first author is also funded by a Postdoctoral Fellowship at the University of Calgary.

References

Baker Jr., G.A. and Graves-Morris, P., 1996, Padé Approximations, Cambridge University Press, Cambridge.

Carcione, J.M., 2007, Wave Fields in Real Media: Wave propagation in anisotropic, anelastic, porous and electromagnetic media, Elsevier, Second ed.

Day, S.M. and Minster, J.B., 1984, Numerical simulation of attenuated wavefields using a Padé approximant method, Geophys. J. R. astr. Soc., 78, 105 -118.

Tal-Ezer, H. and Carcione, J.M. and Kosloff, D., 1990, An accurate and efficient scheme for wave propagation in linear viscoelastic media, Geophysics, 55, 1366 -1379.

Tikhonov, A.N. and Arsenin, V.Y., 1977, Solutions of ill-posed problem, New York: Willey.

Zhang, D. and Cherkaev, E., 2009, Reconstruction of spectral function from effective permittivity of a composite material using rational function approximations: J. Comput. Phys., 228, 5390 -5409.

Zhang, D. and Lamoureux, M.P. and Margarve, G.F., 2009, Estimation of *Q* and phase velocity using the stress-strain relaxation spectrum, CREWES Research Report – Volume 21.

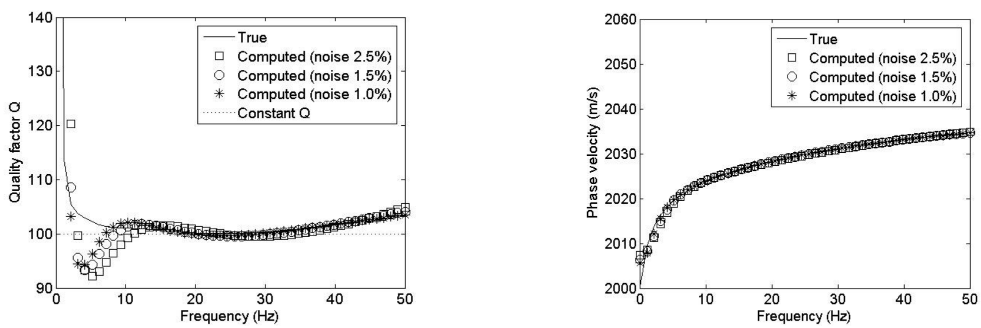

Fig. 1 True and computed *Q*-factor (left) and phase velocity (right) for data with 1.0%, 1.5% and 2.5% noise for the standard linear solid (Zener) model (q = 5 in the inversion algorithm).

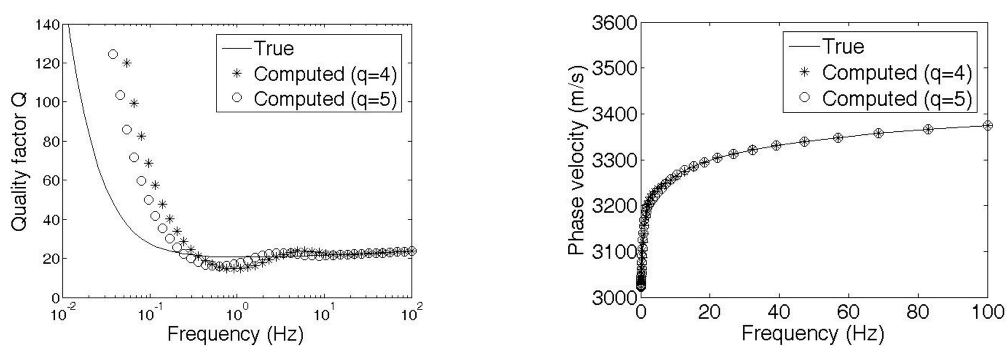

Fig. 2 Calculation of *Q*-factors (left) and phase velocity (right) for nearly constant *Q* =21 model with a continuous spectral measure using different orders of Padé approximation.