Properties of Gabor Operators for Seismic Imaging

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Summary

We characterize the mathematical properties of Gabor transforms and Gabor multipliers, which form an important toolset in seismic data processing techniques incorporating nonstationarity. It is established that the Gabor multiplier is represented as a sum of product and convolution operators. The Gabor mulitplier acts as a "best fit" approximation to pointwise modification of a signal in the Gabor domain. The composition of two Gabor multipliers is approximately a Gabor multiplier, with error term given by a commutator operator. The composition of a Gabor multiplier with a Fourier multiplier is a Gabor multiplier, in certain useful cases. The Gabor multiplier is bounded by the size of the multiplier, in certain useful cases.

Introduction

Seismic imaging occurs in the context of inhomogeneous, anisotropic media, and mathematical modeling of the physical propagation of waves must take into account the local and global variations of parameters that characterize physical properties of the geology. Processing of data should respect these inhomogenities, and typically must be designed in a nonstationary manner, to track the known parameter variations.

A specific implementation of nonstationary filtering that our research groups has been developing uses Gabor transforms and Gabor multipliers in a variety of seismic data processing applications such as spectral deconvolution, depth migration, and reverse time migration. Some recent work on this include (Wards et al., 2008), (Ma and Margrave, 2008), (Ismail, 2008), (Ma and Margrave, 2007a), (Ma and Margrave, 2007b), (Henley and Margrave, 2007), (Montana and Margrave, 2006), (Margrave and Lamoureux, 2006), and (Grossman, 2005). Some foundational references include (Margrave et al., 2003b), (Margrave et al., 2003a) and (Margrave and Lamoureux, 2002). Gabor techniques are an extension of the Fourier methods applied to localized signals, allowing mathematical models of physical material with inhomogeneities.

The key step in the Gabor method is to break up a signal into small, localized packets by multiplying the signal with a window function. Typically, the window is a smooth "bump" function, such as a Gaussian, localized at the point of interest in the signal. The localized packet can then be analyzed or modified using Fourier techniques. This is done for a collection of windows, covering the entire extent of the signal. Finally, all the processed packets are re-assembled into one full, nonstationarily processed signal.

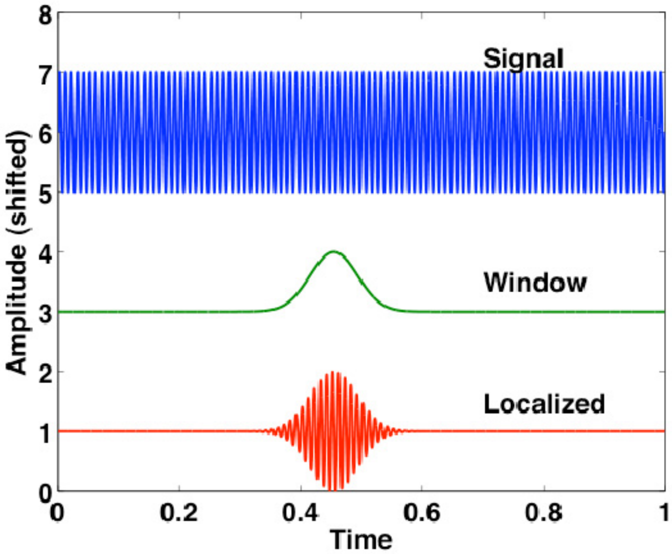

Figure 1: A signal, a window, and the localized packet.

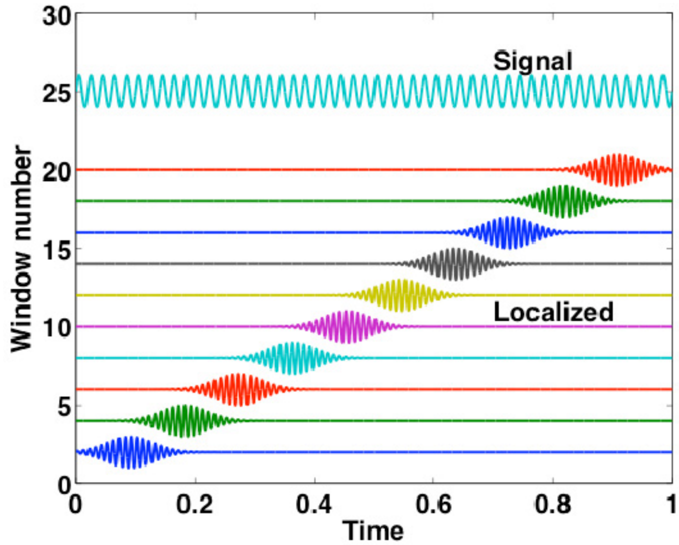

Figure 2: A spread of localized signal packets.

An illustration of this windowing method is presented Figure 1, which shows a sinusoid signal aligned above a bump function. The product of the two is shown in the bottom third of the figure, illustrating a short packet of signal in the middle of the time axis. This packet is a localization of the signal.

More generally, in Figure 2, a signal is decomposed into a sum of several signal packets, illustrated in the figure as a set of 10 packets, localized to different positions on the time axis. For this particular example, the packets are the same except for their location. For a more complex signal, the packets would be different, with each packet capturing the local information of the signal in the interval framed by its window.

A Gabor multiplier acts by modifying the localized packet using a specified Fourier multiplier. Each packet can have a different Fourier multiplier, which allows for nonstationary filtering. The properties of the Gabor multiplier are completely determined by specifying the individual Fourier multipliers that go with it. We refer the reader to the references for examples of constructions of Gabor multipliers used in seismic imaging.

Theory of Gabor transforms and multipliers

The Gabor transform of a signal is given as a localized Fourier transform, so we can define the transform as

$$(Gf)(m,\omega) = F(f \cdot w_m)(\omega),$$

where f is the signal in time (such as a sequence of data points from a single shot record), w_m is a window function for some index m, F is the Fourier transform operator, and ω is the frequency variable.

An inverse to the Gabor transform is obtained by first selecting a set of dual windows $v_1, v_2, ..., v_M$ to go with first windows $w_1, w_2, ..., w_M$ in such a way that their products form a partition of unity,

$$\sum v_m \cdot w_m = 1.$$

The inverse transform is given as the sum (over the index m)

$$(Hg)(t) = \sum v_m(t)\, F^{-1}(g_m)(t),$$

where $g = g(m,\omega)$ is a function of two variables (the window index m, and frequency variable ω), F^{-1} is the inverse Fourier transform, g_m indicates the function of frequency given as $g_m(\omega) = g(m, \omega)$, and t is the time variable for the signal.

The partition of unity condition gives the following result, allowing one to invert the Gabor transform:

Theorem: The composition $HG = I$ is the identity operator.

A bit of calculation show the following as well:

Theorem: The composition $GH = P$ is a self-adjoint projection,

provided the windows w_m, v_m are all real-valued functions.

In particular, the transform H is only a one-sided inverse to the Gabor transform G, which explains why the Gabor theory is not quite as elegant as Fourier theory. However, the fact that $GH = P$ is a projection gives useful results, as shown in the following discussions.

A Gabor multiplier is specified by fixing a function $\alpha = \alpha(m,\omega)$ of two variables, mapping a signal f to its Gabor transform, multiplying by α and then returning to the signal space by inverse transform H. Thus we define the multiplier G_α on a signal f as the composition

$$G_\alpha f = HM_\alpha Gf.$$

Equivalently, the operator G_α is the composition of the Gabor transform G with a pointwise multiplier M_α followed by the inverse transform H. It is remarkable that this nonstationary operator can be expressed as a sum of compositions of multipliers and convolution operators, as in the following:

Theorem: $G_\alpha = \sum M_{vm} C_m M_{wm}$, the sum over m of multipliers and convolutions,

where M_{vm} is multiplication by the window function v_m, C_m is convolution by the function $F^{-1}(\alpha_m)$ and M_{wm} is multiplication by the window function w_m.

The action of the Gabor multiplier is to take the function $f = f(t)$, create its representation in the Gabor domain as $g = g(m,\omega)$, modify it to a new function $\alpha g = \alpha(m,\omega)g(m,\omega)$, and map that back to some new signal $f_1 = G_\alpha f$. It would be wonderful if the result f_1 was represented in the Gabor domain as exactly the function αg. It is not. However, it is close: it turns out that f_1 is the unique signal whose Gabor representation is closest as possible to the function αg. This is summarized in the following:

Theorem: $G_\alpha f = \arg\min \| Gf_1 - \alpha g \| = \arg\min \| Gf_1 - M_\alpha Gf \|$,

where the minimization is taken over argument signal f_1. and the norm is the energy norm in the Gabor space of functions of two variables. This result follows from the observation that GH is a projection.

A related calculation shows that the error between the Gabor representation of $G_\alpha f$ and the "expected" function $\alpha g = M_\alpha Gf$ can be determined exactly, as stated in the following:

Theorem: $\|G\, G_\alpha f - M_\alpha Gf \| = \| [GH,M_\alpha]Gf \|$,

where $[GH,M_\alpha] = GHM_\alpha - M_\alpha GH$ is a commutator of operators. Thus for some choices of multiplier α, the error can be very small -- precisely when M_α nearly commutes with the projection GH.

A similar calculation, relying on the fact the GH is a projection, shows the composition of two Gabor multipliers is nearly a Gabor multiplier. In fact G_α composed with G_β is close to the operator $G_{\alpha\beta}$. The error is also given by a commutator, as summarized in the following:

Theorem: $G_{\alpha\beta} - G_\alpha G_\beta. = H[GH,M_\alpha]M_\beta G = HM_\alpha [M_\beta,GH]G$.

Once again, for certain choices of multipliers and/or windows, these commutators could be small, or zero, so the error term may vanish.

For instance, replacing the Gabor multiplier with the usual stationary Fourier multiplier F_α , where $\alpha = \alpha(\omega)$ is now a function only of the one variable, we obtain

Theorem: $G_{\alpha\beta} = F_\alpha G_\beta$, provided the windows $v_m = 1$ for all m.

Similarly,

Theorem: $G_{\alpha\beta} = G_\beta F_\alpha$, provided the windows $w_m = 1$ for all m.

This result has been used extensively in Gabor decon, where the stationary seismic source, represented by operator F_α , is acted on by a nonstationary Q filter, represented by operator G_β , as discussed in Margrave et al., 2002, 2003a, 2003b, 2006.

A final result concerns the operator norm of the Gabor multiplier, which is restricted by the size of the multiplier α. In certain cases, we can be very specific, as in the following:

Theorem: $\| G_\alpha \| \leq \max | \alpha(m,\omega) |$, where the max is taken over variables m,ω.

This holds in the symmetric case, where the duals windows $v_m = w_m$ match the originals. This bound is important for stability in wavefield extrapolation. In practice, the symmetric window condition can often be dropped, and reasonable stability is preserved.

Conclusions

We have established several mathematical properties of Gabor transforms and Gabor multipliers which are essential to their use as nonstationary data processing tools. We refer the reader to the references for examples of their use, where the desired properties had been conjectured but not yet established.

Acknowledgements

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