Phase-Ray Maslov Summation:
A Complete Finite-Frequency Ray Method

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Fresnel interference and inhomogeneity scattering are two fundamental processes that affect finite-frequency wave propagation in inhomogeneous media. Neglecting the effects of these processes in its formulation, asymptotic ray theory (ART) is essentially an infinite-frequency ray method. It becomes singular near caustics and fails to model a host of finite-frequency wave phenomena important for accurate seismic modeling and imaging. Several finite-frequency ray methods have been developed to overcome the limitations of ART, and they can be divided into two distinct groups: the ray summation methods and the phase-ray method. The ray-summation methods such as the Maslov and Gaussian-beam methods can model the wave phenomena arising from Fresnel interference, but remain unable to account for inhomogeneity scattering. The phase-ray method, on the other hand, can model scattering-related wave phenomena, but ignores the contributions from neighboring non-geometric rays. Thus, both ray-summation and phase-ray methods are only partially successful in modeling finite-frequency wave propagation in inhomogeneous media. I describe in this study a ray method which combines the advantages of both ray-summation and phase-ray methods. By expanding a wavefield into a Maslov summation of phase-ray solutions, the new method can now model the finite-frequency wave phenomena arising from both inhomogeneity scattering and Fresnel interference, and has no singularity problem. It is thus a complete finite-frequency ray method that provides a powerful tool for accurate seismic modeling and imaging in general inhomogeneous media.

Introduction

Asymptotic ray theory (ART) plays an important role in seismic modeling and imaging. It is, for example, a key element of the Kirchhoff migration algorithm which has been a major tool over the past decade for prestack depth imaging. ART has the advantage of simplicity in that a single ray is used to determine the traveltime and amplitude of a particular arrival. In assuming that wave energy propagates along a single ray connecting the source and receiver, however, ART neglects the contributions from both Fresnel interference and inhomogeneity scattering, two fundamental processes that affect finite-frequency wave propagation in inhomogeneous media. Thus, ART is essentially an infinite-frequency ray method (Zhu and Chun, 1994). It becomes singular near caustics and fails to model a host of finite-frequency wave phenomena such as diffractions and frequency-dependent subsurface illumination.

A number of finite-frequency ray methods have been developed to overcome the limitations of ART in modeling Fresnel interference and inhomogeneity scattering. These methods can be divided into two distinct groups: the ray summation methods and the phase-ray method. By expanding a wavefield as a summation of ART solutions, the ray-summation methods such as Maslov and Gaussian-beam methods can model the finite-frequency wave phenomena arising from Fresnel interference between neighboring rays (Chapman and Drummond, 1982; Cerveny et al., 1982), but remain unable to account for the inhomogeneity scattering effects. Based on a phase eikonal equation, on the other hand, the phase-ray method is capable of modeling scattering-related wave phenomena (Zhu and Chun, 1994), but ignores the contribution from neighboring non-geometric rays. Thus, both ray summation and phase-ray methods are only partially successful in modeling finite-frequency wave propagation in general inhomogeneous media. The purpose of this study is to combine the phase-ray method with Maslov summation theory to formulate a complete finite-frequency method that can model both Fresnel interference and inhomogeneity scattering. In the following sections, I first summarize the ART and ART-based Maslov summation methods and then describe their generalization to phase-ray and phase-ray-based Maslov methods.

ART and ART-Based Maslov Summation

ART and Maslov theory have been described in detail, for example, by Chapman and Drummond (1982). I summarize here only the results that are to be generalized in the next section.

ART solution Consider the wave equation in the frequency domain:

$$\nabla^2\Psi - \frac{\omega^2}{V^2(\mathbf{x})}\Psi = 0 \cdot \tag{1}$$

ART seeks an approximate solution to the wave equation in the form of ray series:

$$\Psi(\mathbf{x},\omega) = \sum_{n=0}^{\infty}\frac{A^{(n)}(\mathbf{x})}{(-i\omega)^n}e^{i\omega\tau(\mathbf{x})}, \tag{2}$$

where $A^{(n)}$ is the amplitude of the nth-order term of the series, and $\tau(\mathbf{x})$ the traveltime along the ray. Substituting the solution into the wave equation and retaining only the zero-order terms in the resulting expansion yields the classic eikonal equation

$$(\nabla\tau)^2 = V^{-2}, \tag{3}$$

and the amplitude equation

$$A^{(0)}(\mathbf{x}) = \frac{A^{(0)}(\mathbf{x}_0)}{E(\mathbf{x})}\frac{V(\mathbf{x})}{V(\mathbf{x}_0)}, \tag{4}$$

where $\mathbf{x}_0$ is the source point. The geometric spreading function E in equation 4 is determined by

$$E(\mathbf{x}) = \left[J(\mathbf{x})/J(\mathbf{x}_0)\right]^{1/2}, \tag{5}$$

where $J(\mathbf{x})$ is the Jacobian associated with the transformation between the Cartesian and ray coordinates. The zero-order ART solution given in equations 2, 3, and 4 thus becomes

$$\Psi(\mathbf{x},\omega) = \frac{A^{(0)}(\mathbf{x}_0)}{|E(\mathbf{x})|}\frac{V(\mathbf{x})}{V(\mathbf{x}_0)}\exp[i\omega\tau(\mathbf{x}) - i\pi\,\mathrm{sgn}(\omega)\sigma/2], \tag{6}$$

where KMAH index σ takes into account the accumulated phase shift due to the sign changes in $E(\mathbf{x})$ along the ray (Chapman and Drummond, 1982).

ART-based Maslov summation ART solution 6 assumes that wavefield at receiver point $\mathbf{x}$ is determined by the wavefront information along a single geometric ray connecting the source to this receiver point. This is exact only for infinite frequency because, for finite frequency, the wavefield at $\mathbf{x}$ is the interfering result of a bundle of rays arriving within the Fresnel region around $\mathbf{x}$. Neglecting the contributions of these neighboring rays can result in significant inaccuracy when the wavefront around $\mathbf{x}$ is not sufficiently smooth. To take into account of this wave interfering process, Maslov method represents the wavefield at $\mathbf{x}$ as a summation of neighboring ART rays (Chapman and Drummond, 1982):

$$\Psi(\mathbf{x},\omega) = \left(\frac{i\omega}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} B(p_1, x_2, x_3) e^{i\omega\theta(p_1,\mathbf{x})} dp_1 \,, \tag{7}$$

where $p_1 = \partial\tau / \partial x_1$ is the horizontal component of the slowness vector $\mathbf{p} = \nabla\tau$. The Maslov amplitude $B(p_1, x_2, x_3)$ and Maslov phase $\theta(p_1, \mathbf{x})$ for each ray arrival in summation 7 can be calculated from the ART amplitude and traveltime given, respectively, in equations 4 and 3 (e.g., Chapman and Drummond, 1982).

Phase Ray and Phase-Ray-Based Maslov Summation

Although the ART-based Maslov summation method takes into account the contribution of neighboring rays and is more accurate than ART in modeling finite-frequency wavefield, it is still unable to model wave phenomena related to inhomogeneity scattering. That both ART and the ray summation methods fail to address the scattering-related phenomena is not a coincidence. All these methods inherit, from the classic eikonal equation 3, a simplifying assumption, namely, signals of all frequencies propagate at a single speed and along a single ray path. This amounts to neglecting the scattering effects due to medium inhomogeneities as these effects in general induce dispersion and frequency-dependent propagation paths.

Phase-ray solution To remove the simplifying assumption underlying ART and ART-based ray summation methods, the phase–ray method seeks a ray solution to wave equation 1 in the form of

$$\Psi(\mathbf{x},\omega) = A(\mathbf{x},\omega) e^{i\omega\tau(\mathbf{x},\omega)} \,. \tag{8}$$

The ray solution 8 is more general than that in 2 because its amplitude $A(\mathbf{x},\omega)$ and traveltime $\tau(\mathbf{x},\omega)$ are functions of frequency. Substituting 8 into wave equation 1 and equating the real and imaginary part leads to the phase eikonal equation (Zhu, 1988)

$$(\nabla\tau)^2 = V^{-2} + \omega^{-2}(\nabla^2\gamma / \gamma) \,, \tag{9}$$

and the amplitude equation

$$A(\mathbf{x},\omega) = \frac{A(\mathbf{x}_0,\omega)}{E(\mathbf{x},\omega)} \frac{v(\mathbf{x},\omega)}{v(\mathbf{x}_0,\omega)} \,, \tag{10}$$

where $v(\mathbf{x},\omega) = 1/\left|\nabla\tau(\mathbf{x},\omega)\right|$ is the phase velocity, and the expression for determining $E(\mathbf{x},\omega)$ is identical to that in equation 5, except that the Jacobians are now frequency-dependent. The divergence coefficient γ in equation 9 is given by

$$\gamma(\mathbf{x},\omega) = \frac{1}{E(\mathbf{x},\omega)} \frac{v(\mathbf{x},\omega)}{v(\mathbf{x}_0,\omega)} \,. \tag{11}$$

The phase eikonal equation 9 differs from the classic eikonal equation 3 in that it includes on its right-hand side a frequency-dependent second term. It can be shown using the Born approximation that this term embodies the scattering effects due to velocity gradients and geometric focusing (Zhu, 1988). Retaining this scattering term removes the simplifying assumption of the previous ray methods that signals at all frequencies propagate at a single speed and along a single ray path. In other words, it now allows signals at different frequencies to propagate at different phase velocities and along different ray paths. The phase-ray method thus provides a simple and elegant approach for modeling wave scattering in inhomogeneous media.

The scattering term also introduces signal smoothing and results in a natural removal of the singularity of ART near a caustic (Zhu and Chun, 1994). It is clear from equation 9 that inhomogeneity scattering is a finite-frequency phenomenon. Only at infinite frequency, does the scattering term vanish from equation 9, the classic eikonal equation becomes exact, and the ray bundle connecting a source-receiver pair deflates into a single ray.

The derivation leading to the phase-ray solution given in equations 8, 9 and 10 has been accomplished assuming no approximation. The solution thus represents an exact ray formulation. Except for a few velocity functions, however, exact solutions to the eikonal equation 9 are in general unattainable. Zhu (1988) has shown that equation 9 can be solved by asymptotic expansion, and for the far field, its second-order approximation takes the form

$$(\nabla\tau)^2 = V^{-2} + \omega^{-2}\left[\frac{3}{4}\left(\frac{\nabla P}{p}\right) - \frac{1}{2}\frac{\nabla^2 P}{P} + \frac{\nabla P \bullet \nabla E}{PE}\right], \qquad (12)$$

where slowness $P = 1/V$. The phase-ray solution given in equations 8, 10, and 12 has an estimated accuracy of $O(\omega^{-2})$, two orders of magnitude more accurate than that of $O(\omega^{0})$ for ART and ART-based ray summation methods. It has been successfully used to model a number of scattering-induced wave phenomena such as partial reflections from a gradient zone (Zhu and Chun, 1994).

Phase-ray-based Maslov summation Although more accurate than ART and ART-based ray summation methods, the phase-ray solution cannot adequately model wave phenomena arising from Fresnel interference. At a given frequency, the phase-ray solution is evaluated along a single ray in the same manner as the ART solution except that its eikonal is determined by phase velocity $v(\mathbf{x},\omega)$ instead of medium velocity $V(\mathbf{x})$. Thus, similar to ART, the phase-ray method ignores the contributions from the neighboring rays at the given frequency, and can result in significant inaccuracy when the wavefront at that frequency is not sufficiently smooth around the receiver point $\mathbf{x}$. This limitation can be overcome by expanding a wavefield at each frequency as a summation of neighboring phase rays. This can be accomplished in the same manner as that described in the previous section. The only difference is that the summation must now be carried out at each frequency on a different wavefront, rather than on a single wavefront for all frequencies as for the ART-based Maslov summation. This leads to the phase-ray Maslov summation solution:

$$\Psi(\mathbf{x},\omega) = \left(\frac{i\omega}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} B(\mathbf{y},\omega) e^{i\omega\theta(p_1,\mathbf{x},\omega)} dp_1(\omega). \qquad (13)$$

Similar to the ART-based Maslov summation method, the amplitude $B(\mathbf{y},\omega)$ and phase function $\theta(p_1,\mathbf{x},\omega)$ in equation 13 can be calculated from the phase-ray amplitude 10 and traveltime 12 with the formula given by Chapman and Drummond (1982). The phase-ray Maslov summation solution in 13 can now model both Fresnel interference and inhomogeneity scattering, and is valid at caustics. It thus provides a powerful tool for accurate modeling finite-frequency wave propagation in general inhomogeneous media.

Conclusions

ART is essentially an infinite-frequency ray method. It becomes singular and fails to model the finite-frequency wave phenomena arising from both Fresnel interference and inhomogeneity scattering. To overcome the limitations of ART, I have formulated in this study a new ray method by combining the advantages of the ray-summation and phase-ray methods. By expanding a wavefield as a Maslov summation of phase-ray solutions, the newly formulated phase-ray Maslov summation method can now model both Fresnel interference and inhomogeneity scattering, and has no singularity problem. It is thus a complete finite-frequency ray method and provides a powerful tool for accurate seismic modeling and imaging in general inhomogeneous media

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