

Estimation of Arithmetic Permeability from a Semi-Log Porosity/Log₁₀ Permeability Regression*

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Abstract

Permeability in geologic model cells is usually estimated using a semi-log Porosity: Permeability transform based on core plug data, because permeability typically has a log-normal distribution (i.e. the logarithms of permeability form a Gaussian distribution). Pore networks in rocks are neither completely random nor perfectly correlated, with reality being somewhere between these extremes. Theoretically, a geometric permeability average approximates flow through a random network of pores, and an arithmetic permeability average approximates flow through a perfectly correlated network of pores. The results of effective property simulation in ultra-fine scaled models support the history-matching observation that effective permeability is rarely (if ever) equal to the geometric mean regression. An attempt was made at an empirical solution for “boost” that separates the arithmetic regression from the geometric regression. For data with a log-normal distribution, there is a simple relationship between the geometric average (center-of-cloud) Porosity: Log₁₀ Permeability transform and the arithmetic transform.

Introduction

Permeability in geologic model cells is usually estimated using a semi-log Porosity: Permeability transform based on core plug data; because permeability typically has a log-normal distribution (i.e. the logarithms of permeability form a Gaussian distribution). Porosity: Log₁₀ Perm regressions are expressed in the form:

$$(1) \log_{10}K_g = (\text{Slope} * \text{Porosity}) + \text{Intercept}$$

By definition, this is a geometric mean regression because it goes through the center of a log₁₀ Permeability cloud, and therefore seeks the average of logarithms. However, when flow simulation is run on these models, we routinely find that we need to multiply these permeabilities by factors between 3 and 7 in order to achieve a history match. Why?

Discussion

Pore networks in rocks are neither completely random nor perfectly correlated, with reality being somewhere between these extremes (Figure 1). Theoretically, a geometric permeability average approximates flow through a random network of pores, and an arithmetic permeability average approximates flow through a perfectly correlated network of pores. The geometric mean is therefore the theoretical minimum possible horizontal permeability, and the arithmetic average is the theoretical maximum possible horizontal permeability. Recall that history matching teaches us that the true permeability commonly must be several times the geometric average minimum.

The results of effective property simulation in ultra-fine scaled models (Figure 2) support the history-matching observation that effective permeability is rarely (if ever) equal to the geometric mean regression. If we plot the upscaled flow-based scale-averaged (FBSA) effective permeabilities of larger-scaled cell aggregates on the same graph with the core-plug scale laboratory analysis data used to populate the ultra-fine scaled models (Figure 3), they consistently plot in a cloud that is narrower and centered on a higher average permeability. The amount of effective permeability “boost” was observed to vary proportionally with the width of the input core plug cloud. We investigated this observation first using single-iteration simulation (Figure 4) and later using a Monte-Carlo technique (Figure 6).

Method

A piecewise arithmetic transform can be made by 1) dividing the core plugs into porosity bins, 2) calculating the arithmetic mean of the permeability values in each bin, 3) plotting the bin averages on a Porosity:Log₁₀ Permeability crossplot, and 4) connecting the points with line segments (Figure 5). The result is a slightly crooked line located a nearly uniform distance above the geometric regression line.

Encouraged by this graphical display, an attempt was made at an empirical solution for “boost” that separates the arithmetic regression from the geometric regression (Figure 6). A Monte Carlo approach calculated arithmetic means on hundreds of log-normal distributions. This approach provided an excellent-fit regression for calculating the “boost” between the arithmetic mean and the geometric mean. This near-perfect regression strongly suggested a theoretical solution probably existed.

A mathematical proof verifies that the “boost” of the arithmetic mean over the geometric mean is rigorously defined as a function of the standard deviation of the porosity-permeability point cloud residual around the geometric regression (Figure 7). The equation describing the “boost” is:

$$(2) \log_{10}K_a = \log_{10}K_g + 1.1513 * STD^2$$

where:

K_a = Arithmetic Average

K_g = Geometric Average

STD = Standard Deviation of the permeability residual with respect to the trend, in Log₁₀ Perm units

To apply this technique, 1) calculate the \log_{10} of perm, 2) find the best-fit semi-log regression through the porosity: \log_{10} permeability point cloud, 3) calculate the difference in \log_{10} permeability units between each point and that regression (the residual), 4) calculate the standard deviation of the \log_{10} residual, 5) square the standard deviation and multiply it by 1.1513, and 6) add this amount to the intercept of the geometric average equation (1), which is equivalent to a permeability multiplier because we are adding logarithms.

Conclusion

For data with a log-normal distribution, there is a simple relationship between the geometric average (center-of-cloud) Porosity: \log_{10} Permeability transform and the arithmetic transform. The difference between the geometric and arithmetic transforms is solely a function of the width of the \log_{10} Permeability data cloud around the semi-log trend line expressed in logarithmic decades. The semi-log arithmetic regression estimate represents the maximum possible horizontal permeability. In most porous media, the true effective permeability will fall somewhere between the geometric regression and the arithmetic regression. If simulation history-matching suggests a need for permeabilities greater than the arithmetic average, we should consider the possibility that there are features in the reservoir for which we have not accounted, such as fracturing or unrecognized “Super Permeable” sub-facies. Similarly, the semi-log harmonic regression estimate is the theoretical minimum vertical permeability. The harmonic mean is symmetrically located on the other side of the geometric mean from the arithmetic mean. Again, the true reservoir permeability will fall somewhere between the geometric mean and the harmonic mean.

It would be extremely unusual for a reservoir to have a horizontal permeability equal to the geometric mean, because that is the theoretical solution for a very random distribution of permeabilities at the reservoir model cell scale. Most rocks have layering, and the effect of layering will be to shift effective permeabilities away from the geometric mean. This is why models built with semi-log center-of-cloud porosity:permeability regressions typically require horizontal permeability multipliers between 3 and 7 and why vertical permeabilities require $K_v:K_h$ multipliers between 1/3 and 1/7 to achieve a history match. The Equation 2 “boost” is an estimate of the maximum allowed permeability multiplier!

We must also remember that effective property simulation tells us that the width of the effective permeability point cloud needs to be reduced. This should be intuitively obvious since it is extremely unlikely that any large-scale cell will have effective permeabilities as high as individual outlier core plugs. If the full-scale reservoir model cells are populated using the “boosted” arithmetic regression, but still using the original standard deviation of the core plugs, the model cells will be populated with permeabilities outside the range of the core plug data, which is nonsense. In reality, the true effective permeability of the reservoir requires a standard deviation significantly smaller than the original core plug standard deviation. When the “boost” of the arithmetic regression over the geometric regression is known, it is not difficult to calculate the maximum allowable standard deviation that will produce values only within the original core plug point cloud, but in reality, the true standard deviation should be even smaller.

Acknowledgement

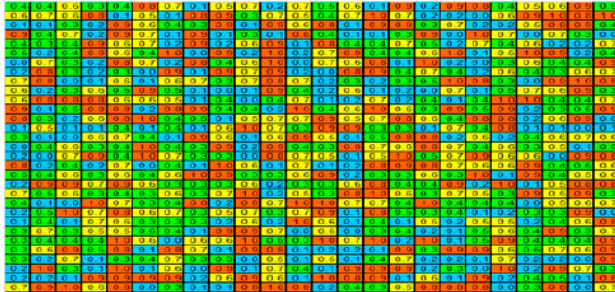
I thank Max Deffenbaugh for developing the rigorous mathematical proofs for the arithmetic and harmonic boost calculations. Max’s work preceded the discovery of the Ababou and Wood paper by several months.

Reference Cited

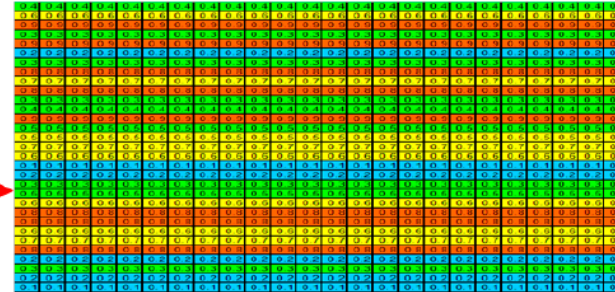
Ababou, R., and E.F. Wood, 1990, Comment on 'Effective Groundwater Model Parameter Values: Influence of Spatial Variability of Hydraulic Conductivity, Leakance, and Recharge' by J.J. Gomez-Hernandez and S.M. Gorelick: Water Resources Research, v. 26/8, p. 1843-1846.

Theoretical Approximations for Effective Permeability

Random Correlation



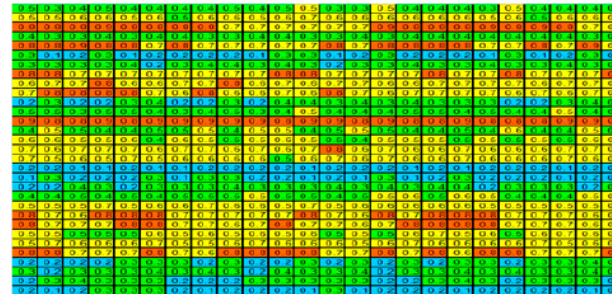
Perfect Horizontal Correlation



- **Geometric Average** approximates flow in any direction

- **Arithmetic Average** approximates *Horizontal Flow*
 - theoretical maximum possible: best perm aligned
- **Harmonic Average** approximates *Vertical Flow*
 - theoretical minimum possible: worst perm aligned

Reality will be somewhere between these extremes



Where in between depends on the cell size in combination with the continuity of beds and the correlation of permeability within individual beds

Figure 1. Maximum and Minimum Effective Permeabilities. The harmonic mean permeability is generally considered to represent the minimum possible vertical permeability because it assumes the lowest permeability bed will control flow (i.e. a thin, extensive shale bed). The geometric mean permeability is thought to represent the minimum possible effective horizontal permeability of a comprehensive sample group because it assumes permeability is randomly connected and so all flow is bottlenecked by the median perm. The arithmetic mean is considered the maximum possible effective horizontal permeability, since it represents a rock in which all samples with equal permeability are perfectly connected laterally.

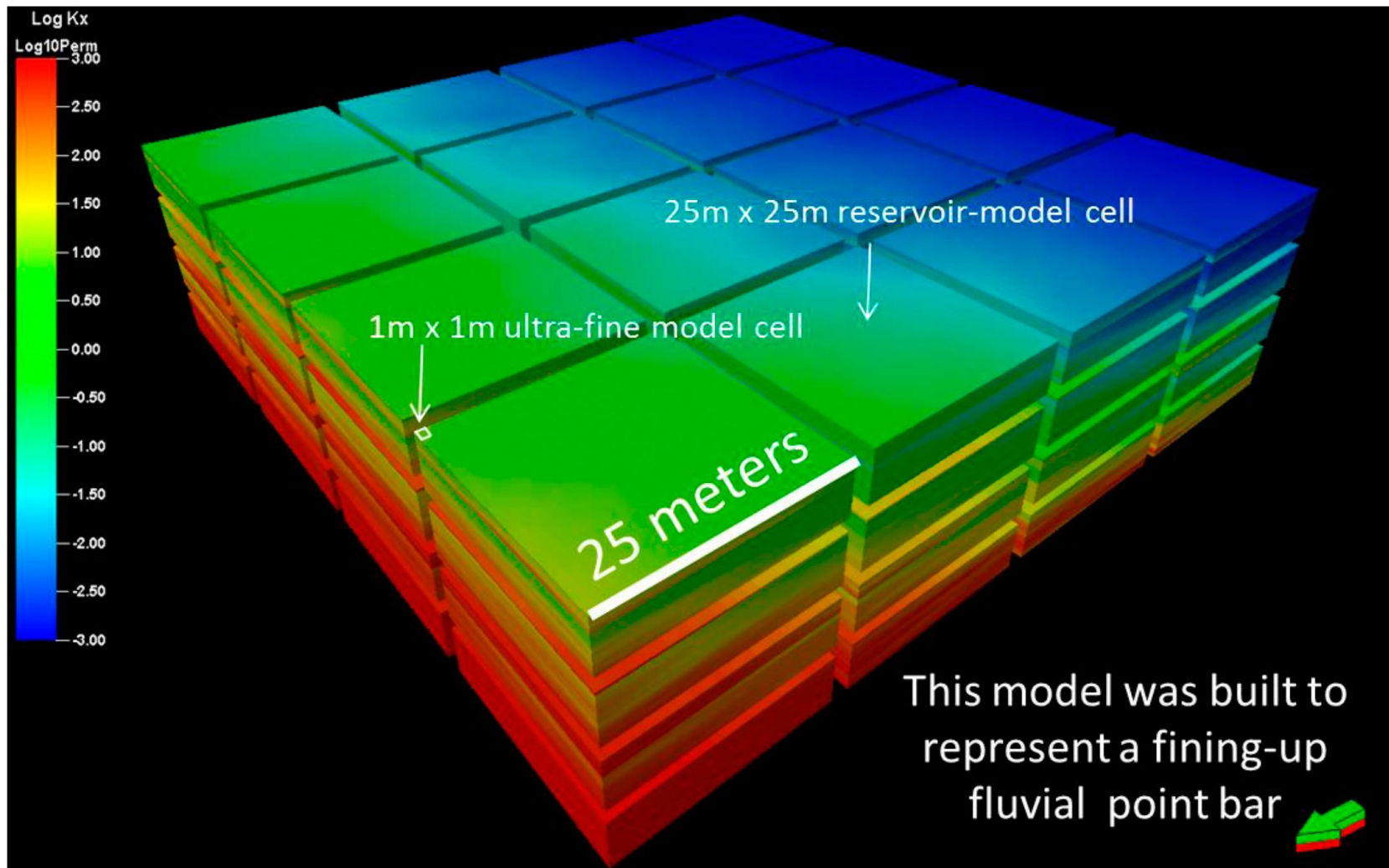


Figure 2. Effective Property Characterization. Multiple ultra-fine scaled models ($\sim 1 \times 1 \times 0.1$ m cells) of porosity and permeability were populated using core plug measurements. In the ultra-fine scaled models, rock types, data ranges, variograms, Porosity: Permeability transforms, and standard deviations were varied to realistically mimic several depositional environments. The example above represents a sector of a fining-up point bar. These sector models are divided into volumes that match the dimensions of the larger-scale reservoir model that will be populated with effective properties ($25 \times 25 \times 1.0$ m aggregates of ultra-fine cells in the example above). These sub-volumes are then flow-based scale-averaged (FBSA) to obtain effective permeabilities in three dimensions. Additional volumetric characterizations such as average porosity, facies proportions, and porosity-weighted irreducible water saturation are also calculated for each sample's sub-volume. These statistics are accumulated in a database for future use in populating coarser-scaled reservoir models.

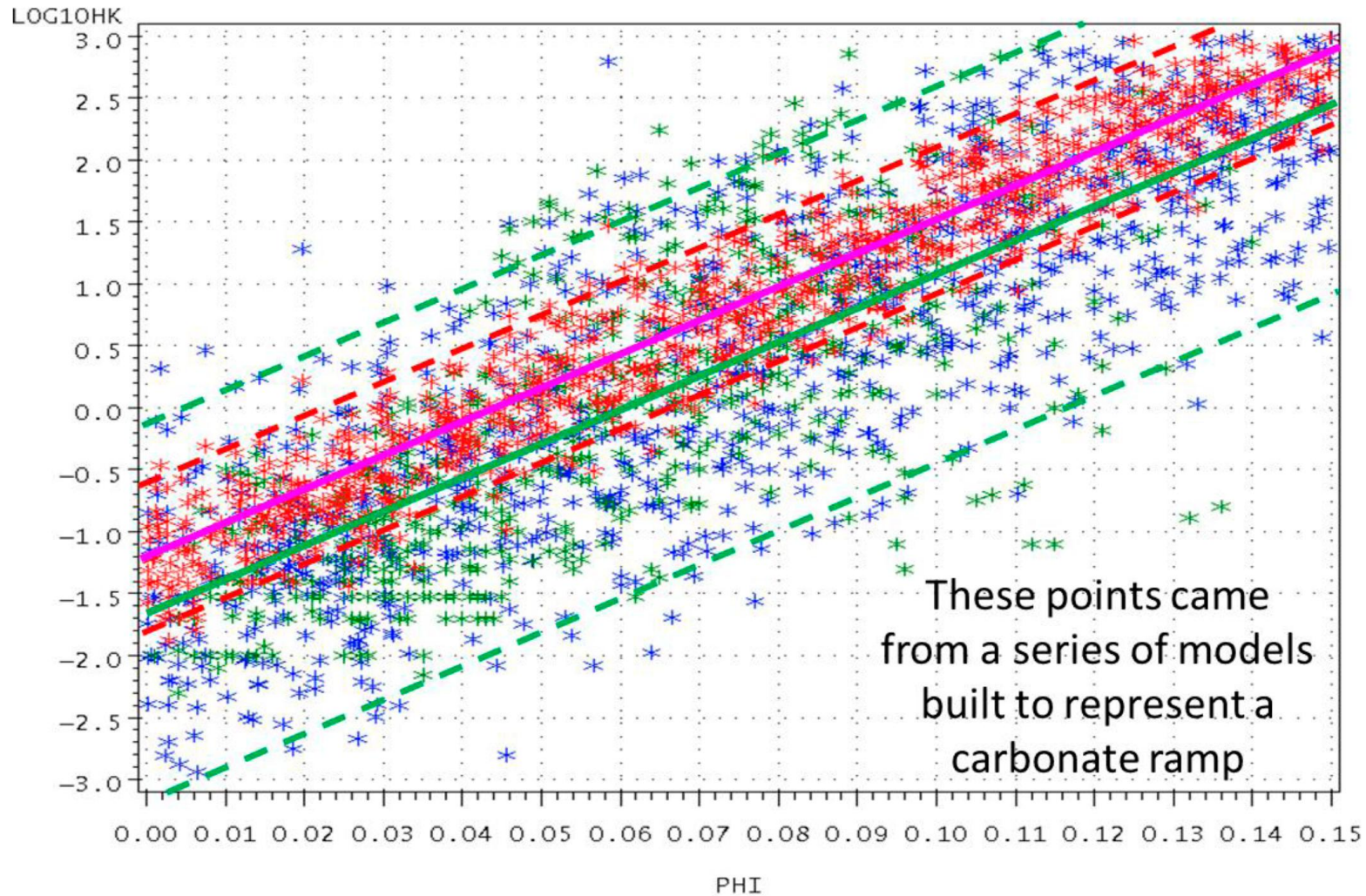


Figure 3. Comparison of Effective Properties. Core plug (green) and ultra-fine scaled model (blue) are plotted with FBSA upscaled effective properties of reservoir model sized aggregates of ultra-fine scaled cells (red) in this Porosity/ Log_{10} Permeability crossplot. The solid green line is the core plug semi-log regression, and the dashed green lines indicate the approximate 90% boundaries of the core plug data. The magenta line is the semi-log regression for the FBSA-upscaled average porosity vs. effective horizontal permeability, and the dashed red lines are the approximate 90% envelope. Note that the red upscaled property points form a narrower cloud centered half an order of magnitude above the center of the green core plug points. For all models with realistic heterogeneity, the upscaled values were always higher than the center of the Porosity: Log_{10} Permeability cloud. However, the distances of the upscaled cell values above the center of the point clouds was observed to vary, with the gap being greater when the core plug point cloud standard deviation is higher.

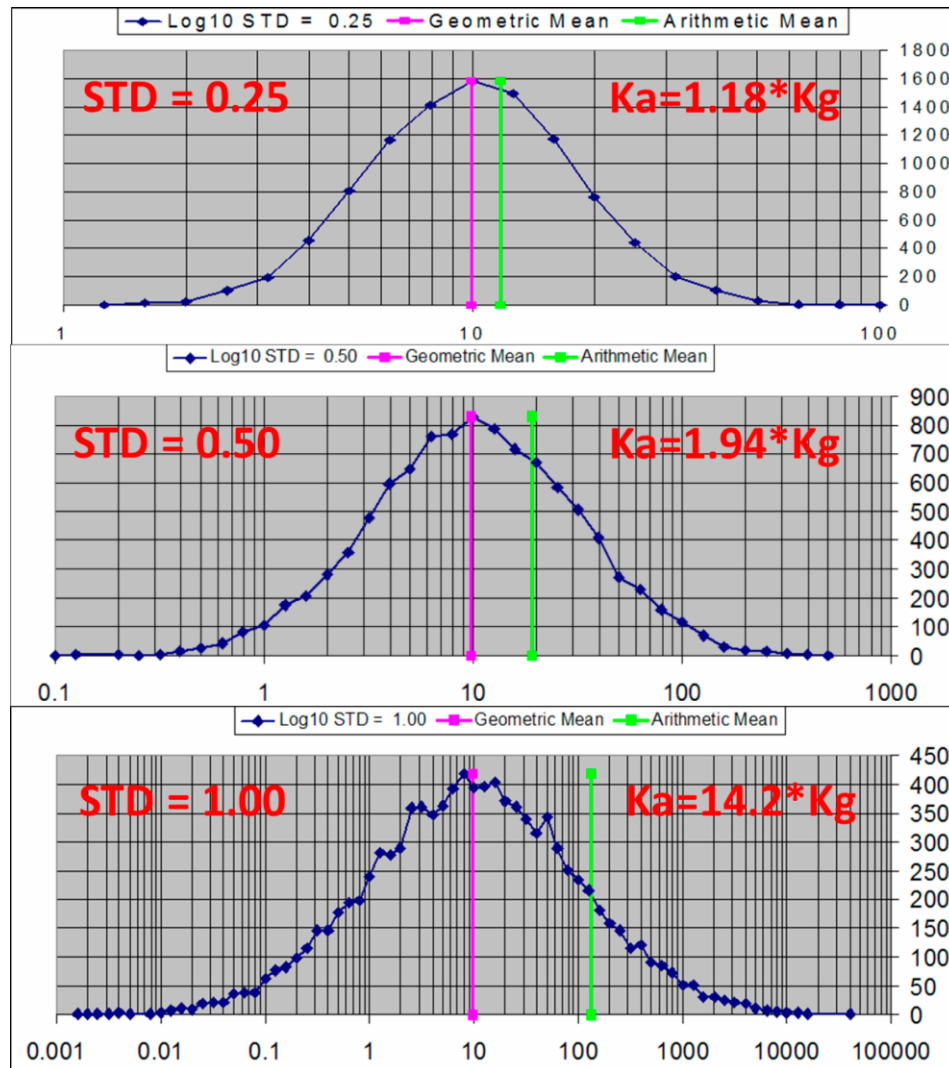


Figure 4. Comparison of Log-Normal Arithmetic and Geometric Means. These three histograms illustrate the changing relationship of geometric mean to arithmetic mean across a range of standard deviations. All of the graphs use the same geometric mean (10) and 10000 randomly generated log-normally distributed samples. Note that the separation between the Geometric Mean (pink) and Arithmetic Mean (green) increases as standard deviation increases. The number of logarithmic decades of uplift is only sensitive to the standard deviation of the log-normal distribution. Changing the geometric mean value has no effect on the relative amount of arithmetic mean uplift in \log_{10} units. This phenomenon occurs because as the standard deviation increases the population below the mean approach asymptotically the lower limit of zero while the population of points above the mean expand exponentially toward infinity.

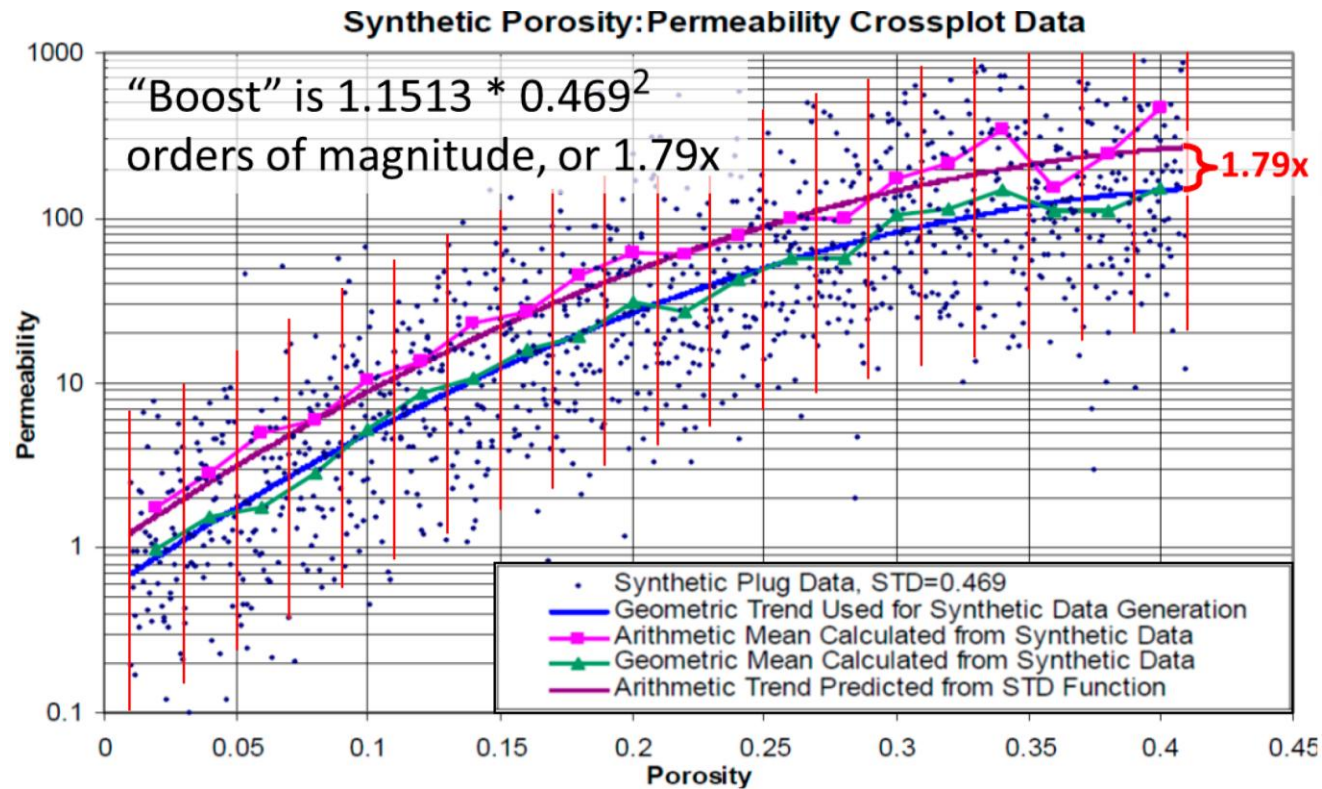


Figure 5. Synthetic Porosity:Permeability Crossplot. This crossplot shows a synthetic Porosity-Permeability dataset (dark blue cloud of points). The semi-log (geometric average) trend line used to create the synthetic point cloud is the blue line. The data was divided into 0.02 porosity fraction bins and then we calculated the geometric averages of permeabilities (green points), and the arithmetic averages of permeability data (pink points). The segmented green line connects the geometric average permeability values of the bins. The segmented pink line connects the arithmetic average permeability values of the bin.

This method works if you have dense enough core plug data, but it is cumbersome to divide the data into bins and calculate the arithmetic averages for each bin, and then create a regression through those points. If the core plug data is sparse, it is likely that the resulting binned arithmetic average points will have anomalies that skew the regression (note the zigzags in the lines connecting the bin averages caused by stochastic chatter in the synthetically derived core plug data bins).

We first investigated an empirically based method to determine the position of the arithmetic average (segmented pink line) using only a semi-log (geometric) regression line and a measure of the permeability scatter was investigated. This evolved later into a theory-based solution. The theory-based arithmetic regression (burgundy line) is exactly parallel to the blue line, but uplifted somewhat (0.25 orders of magnitude, or 1.79 times). The key is finding the magnitude of the uplift or “boost” and then adding that to the geometric regression equation’s intercept.

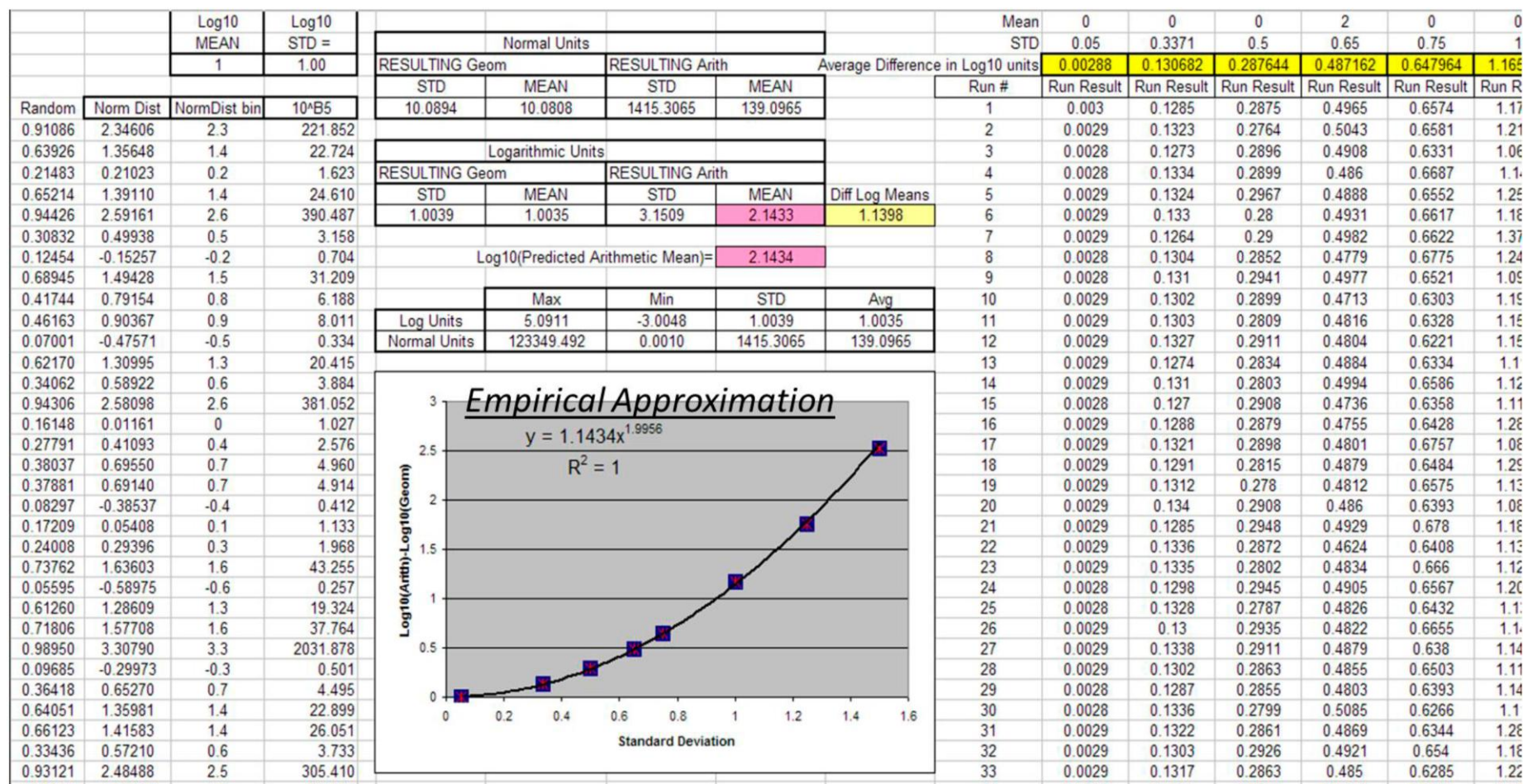


Figure 6. Quest for an Empirical Solution. A Monte Carlo Spreadsheet was used to make randomly generated log-normal distributions with different mean and standard deviation settings. The arithmetic and geometric averages of hundreds of generated distributions were calculated. The result of each iteration was added to a table for graphing. The resulting graph is shown in the center of the spreadsheet. The horizontal axis is standard deviation of the distributions. The vertical axis is the amount of “boost” between the geometric and arithmetic means of the distributions, expressed in \log_{10} units. A perfect fit to the empirical data resulted ($R^2 = 1.00$). This near-perfect empirical relationship led us to seek the underlying mathematical theorem. The empirical regression for the “boost” had an exponent of 1.9956, a value so close to 2 that it seemed likely that 2 was the theoretical value we sought. The standard deviation scalar 1.1434, however, did not immediately suggest any theoretical basis.

Ababou & Wood (1990 p.1844) show that the arithmetic average can be expressed as a function of the geometric mean and sigma-squared (variance).

$$K_A = K_G \exp (+ \sigma_\gamma^2 / 2)$$

$$K_H = K_G \exp (- \sigma_\gamma^2 / 2)$$

where K_A represents the arithmetic average, K_G the geometric average, K_H the harmonic average, and σ_γ is the standard deviation of the residual to the trend in natural logarithm (base e) units.

To convert the Ababou & Wood equations into the form of the empirical fit, we first take the natural logarithm of both sides of the Ababou & Wood equations. Then we convert from natural logarithms (base-e logarithms) to base-10 logarithms by multiplying by $\log_{10}(e) = 2.3026$

Due to the division by 2 in Ababou & Wood , the scalar becomes $2.3026 / 2 = 1.1513$

The equations become:

$$\log_{10}K_a = \log_{10}K_g + 1.1513 * STD^2$$

$$\log_{10}K_h = \log_{10}K_g - 1.1513 * STD^2$$

where:

K_a = Arithmetic Average

K_g = Geometric Average

K_h = Harmonic Average

STD = Standard Deviation of the residual with respect to the trend, in \log_{10} Permeability units

Figure 7. The Theoretical Solution. A clever colleague* provided a mathematical proof that my empirical solution was nearly correct. I later found Ababou and Wood (1990), which I found far easier to comprehend than the mathematical proof. The empirical equation STD scalar of 1.1434 was quite close to the correct theoretical solution of 1.1513. The amount of “boost” between the geometric and arithmetic averages is solely dependent upon the standard deviation of the permeability with respect to the geometric regression. The “boost” is only affected by the standard deviation, not by the mean. The harmonic mean is symmetrically located on the other side of the geometric mean. For the example, application is shown in [Figure 5](#); the poro-permeability cloud has a standard deviation of 0.469 orders of magnitude around the center-of-cloud geometric transform permeability. According to Equation 2, the arithmetic average transform is $0.469^2 * 1.1513 = 0.2532$ orders of magnitude higher than the geometric average. We add this “boost” to the intercept of Equation 1. Because adding logarithms is equivalent to multiplication, this is a permeability multiplication factor of 1.792 times ($10^{0.2532} = 1.792$).