Prewhitening with Matrices

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Summary

Prewhitening, also known as Tikhonov regularization, has always been a standard part of deconvolution, and consequently appears in most seismic processing flows. Usually prewhitening is described as being necessary to ensure stability of the deconvolution filter. In the case of frequency-domain spiking deconvolution, this explanation is clearly justified, since the expression of the filter can contain division by zero if the prewhitening factor is not included.

In time-domain spiking deconvolution, though, the situation is not as clear-cut. Here the filter is obtained by solving normal equations. These equations can be modified in a simple way to include prewhitening; however, unlike the frequency-domain case, it is not intuitively obvious why the modifications should produce a more stable filter.

To this end, it is useful to describe deconvolution and prewhitening in terms of matrix operations. This approach can make it easier to understand how prewhitening stabilizes the filter in the time domain, and how time-domain prewhitening differs from frequency-domain prewhitening. The matrix approach also points to some interesting extensions of time-domain prewhitening, in which autocorrelations are replaced by triple correlations in the normal equations.

Theory

In frequency-domain deconvolution, the Fourier transform of the filter, denoted by $\hat{f}$, is obtained from $\hat{x}$, the Fourier transform of the estimated wavelet, by solving

$$\hat{x}\hat{f} = 1. \tag{1}$$

At each frequency, the solution of (1) is approximated by

$$\hat{f} = \frac{\exp(-i\arg(\hat{x}))}{|\hat{x}| + \epsilon}, \tag{2}$$

where ϵ is the (positive) prewhitening factor. Here, the role of ϵ is clear: it prevents $\hat{\boldsymbol{f}}$ from becoming infinite if $\hat{\boldsymbol{x}}$ has any zero entries.

In time-domain deconvolution, the equivalent of (1) is

$$x * f = d, \tag{3}$$

where $*$ denotes time-domain convolution; x and f are time-domain versions of $\hat{x}$ and $\hat{f}$; and d is the desired output, a spike of unit amplitude followed by $(N_x + N_f - 2)$ zeroes. (N_x is the length of x, and N_f is the length of f.)

To obtain a least-squares estimate of f from (3), we must solve the familiar normal equations

$$\begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_{N_f-1} \\ r_1 & r_0 & r_1 & \cdots & r_{N_f-2} \\ r_2 & r_1 & r_0 & \cdots & r_{N_f-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N_f-1} & r_{N_f-2} & r_{N_f-3} & \cdots & r_0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N_f-1} \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4}$$

where f_j and x_j denote the j^{th} entries of f and x, and r_j denotes the j^{th} lag of the autocorrelation of x,

$$r_j = \sum_l x_l x_{l-j}. \tag{5}$$

The usual way of including prewhitening in time-domain deconvolution is to alter (4) to

$$\begin{bmatrix} r_0(1+\epsilon) & r_1 & r_2 & \cdots & r_{N_f-1} \\ r_1 & r_0(1+\epsilon) & r_1 & \cdots & r_{N_f-2} \\ r_2 & r_1 & r_0(1+\epsilon) & \cdots & r_{N_f-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N_f-1} & r_{N_f-2} & r_{N_f-3} & \cdots & r_0(1+\epsilon) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N_f-1} \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6}$$

In contrast with the frequency-domain case, expression (6) does not give us much intuitive idea of why multiplying r_0 by $(1 + \epsilon)$ should stabilize f. To this end, it is useful to recast (3) as a matrix operation,

$$\mathbf{X}\mathrm{f} = \mathrm{d}, \tag{7}$$

where f and d are considered as column vectors, and **X** is a matrix that represents convolution with f:

$$X = \begin{bmatrix} x_0 & 0 & \cdots & 0 \\ x_1 & x_0 & \ddots & \vdots \\ x_2 & x_1 & \ddots & 0 \\ \vdots & x_2 & \ddots & x_0 \\ \vdots & \vdots & \ddots & x_1 \\ x_{N_x-1} & \vdots & \ddots & x_2 \\ 0 & x_{N_x-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{N_x-1} \end{bmatrix}. \tag{8}$$

This matrix provides a quick way of deriving (4); this is done by multiplying both sides of (7) from the left with X^T, and substituting (8) into the result.

There are two ways of modifying (7-8) to include prewhitening. Both of these lead to (6). The first is to augment an identity matrix, scaled by $\sqrt{\epsilon r_0}$, to the bottom of X. Corresponding zeroes must then be added to the bottom of d in (7). In block matrix notation, this amounts to changing (7) to

$$\begin{bmatrix} X \\ --- \\ \sqrt{\epsilon r_0} I \end{bmatrix} f = \begin{bmatrix} d \\ --- \\ 0 \end{bmatrix}. \tag{9}$$

By expanding the $\boldsymbol{X}$ from (7) into a circulant matrix, one can show that this method of prewhitening has similarities with prewhitening in the frequency domain, as described by equation (2). However, the two approaches turn out not to be identical.

The second way of modifying (7-8) to include prewhitening is to augment a larger identity matrix, again scaled by $\sqrt{\epsilon r_0}$, to the right side of X. This requires additional rows (denoted by q) to be added to f:

$$\left[\begin{array}{c|c} X & \sqrt{\epsilon r_0} I \end{array}\right] \begin{bmatrix} f \\ --- \\ q \end{bmatrix} = d. \tag{10}$$

Considering the columns of this modified X as being frame members, and q as being associated with the least-squares error, makes prewhitening easier to understand from a physical standpoint.

The second modification also points the way to further changes that lead not to expression (6), but to a new expression in which the matrix of autocorrelations from (6) is replaced with a matrix consisting mostly of triple correlations,

$$\begin{bmatrix} r_{0,0} + \epsilon r_0 & r_{1,0} & r_{2,0} & \cdots & r_{(N_f-1),0} \\ r_{1,0} & r_{1,1} + \epsilon r_0 & r_{2,1} & \cdots & r_{(N_f-1),1} \\ r_{2,0} & r_{2,1} & r_{2,2} + \epsilon r_0 & \cdots & r_{(N_f-1),2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{(N_f-1),0} & r_{(N_f-1),1} & r_{(N_f-1),2} & \cdots & r_{(N_f-1),(N_f-1)} + \epsilon r_0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{(N_f-1)} \end{bmatrix} = \begin{bmatrix} p_0 x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{11}$$

Here r_0 has the same meaning as before, and the $r_{j,k}$ denote triple correlations. These are obtained from two time-shifted versions of r, and a third function p,

$$r_{j,k} = \sum_l p_l x_{l-j} x_{l-k} \tag{12}$$

$$\sum_l \frac{1}{p_l} = N_x + N_f - 1. \tag{13}$$

The inclusion of p effectively allows different amounts of prewhitening to be applied at different samples of d. Fixed and data-dependent variants of p are permissible as long as constraint (13) is satisfied. If p is set equal to 1 everywhere, (11) reduces to (6).

In the talk, some examples of deconvolution filters that have been obtained using the triple correlation approach will be compared with similar filters obtained from (6).

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