# *Regime Rays*: Visualizing the Fermat/Snell Loci in Homogeneous Anisotropic Media

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**Summary**

Present understanding and analytical encapsulation of anisotropy kinematics, even in homogeneous anisotropy medium segments which are here the focus, is incomplete/incorrect re the familiar Fermat/Snell ray theory premises [-> *FS-rays*]. The traveltime expression $t_{ON} = t_{OE} = (r_E \bullet n_N)/ v_N (n_N , c_{ij}, \rho)$, attributed to Green (Love 1927, Rudzki 1911), has been deemed essentially a mode-specific plane-wave progression dictate [here $_{O\,,\,N\,,\,E}$ subcripts designate Origin, front-Normal points/directions, Energy-flux arrival- points/directions]. It is moreso a constraint, call it an *ansatz* (starting premise), which for particular homogeneous anisotropic medium models and specified front-normal direction firms direction/progression-speed of *FS-ray* segmentals in part along front-normal direction but, it turns out, with complementing segmentals with maximally two other orientations. *Regime rays* is label for representation loci, when like-oriented segmentals have been re-ordered to single segments, and those segments are then linked.

Through *regime rays* we can discern and quantify significant analytical *FS-rays* detail, the time-fractions and segment lengths plus progression speeds for the three or fewer oriented segments. Those details yield the significant *FS-ray*-specific characterizing velocities for the along-paths velocity distributions $\{v_{APPARENT}, v_{TIME\text{-}AVE}, v_{RMS}, v_{PATH\text{-}MEAN}\}$; also $l_{PATH}$ which are true pathlengths for intangible *FS-ray* loci.

All this comes from overlooked extra prescribed directions/progression-speeds detail in sequenced mode-relevant eikonal equations, relevant for *FS-rays* for given front-normal directions. The *FS-rays* per se, say in medium natural coordinate frame, progressing mode-specific from point-shots $\boldsymbol{r_O}$ at origin to $\boldsymbol{r_E}$-front-points, are anchored to their energy flux line-loci, the 'so-called *group*- or *ray*-velocity representation loci'. Beyond analytical essentials previously communicated I provide here simulations/ visualization for all modes of an orthorhombic standard model. *Regime rays* clarify the long-puzzling detail between paired points on the wave surfaces and their *ansatz*-front-normal velocity representation surfaces.

## From traveltime *ansatz* to *regime rays* models

It is well established that wave propagation disturbances through composites of fine-structured layering and fine-structured heterogeneities broadly, have EMT (effective medium theory, long wavelength regime) progression speeds that are virtually constant beyond a distinct transition zone re wavelength/composites-repetition-thickness ratio (e.g. Macbeth 2002). I suggest that Fermat's principle and Snell's law should apply in such composites also for long wavelength regime *FS-rays*, for progression directions not just orthogonal or parallel to ordered layering or other fine-scale structured heterogeneity. This can be credibly expected, because in the broader physics context Snell's law and Fermat's principle are encompassed within the remarkable '*principle of least action*' of Maupertuis [1746], [also Leibniz, Euler, Hamilton, Feynman,… ]. At any rate, what manifests from this hypothesis agrees with analytical models and kinematics manifestations in diverse experiments. As simplest example take the above pathtime *ansatz*

$t_{ON} = t_{OE} = (\boldsymbol{r}_E \bullet \boldsymbol{n}_N)/\ v_N\ (\boldsymbol{n}_N\ ,\ c_{ij}\ ,\rho\ )$, but reduced to progression paths in just the x.vs.z plane as $t_{ON} = t_{OE} = (x_E \sin\theta + z_E \cos\theta) / v_N$ , with $\theta$ as polar angle re z-axis direction. Then if $x_E > z_E \tan\theta$, the as two segments partitioned equivalent is $t_{ON} = t_{OE} = (z_E /\cos\theta)/ v_N + (x - z_E \tan\theta)/(v_N /\sin\theta)$. First segment has length $l_1 = (z_E /\cos\theta)$ with progression speed $v_1 = v_N$ , and the second $l_2 = (x - z_E \tan\theta)$ with speed $v_2 = (v_N /\sin\theta) > v_N$ . Transition $v_1 = v_N$ to $v_2 = v_N /\sin\theta$ conforms with Snell's law, front-normal oriented $l_1$ of *regime ray* in y=0 plane refracting into x-direction oriented $l_2$ , ending at front point [$x_E$ $z_E$ ].

Else if $z_E \tan\theta > x_E$ , the *ansatz* transforms to $t_{ON} = t_{OE} = (x_E /\sin\theta)/ v_N + (z_E - x_E / \tan\theta)/(v_N /\cos\theta)$, with $l_1 = (x_E /\sin\theta)$ and speed $v_1 = v_N$ front-normal oriented. Then Snell's law conforming refraction produces z-directed $l_2 = (z_E - x_E / \tan\theta)$ with $v_2 = (v_N /\cos\theta) > v_N$ , ending at [$x_E$ $z_E$ ] front-point.

The common pathtime *ansatz* expression has been reframed in two ways, depending on specific association detail between front-normal orientations and for given medium parameters consequent energy flux loci (so-called ray- or group-velocity representation loci), both with two-segmented *regime ray* components.

The general *ansatz* can/must be elaborated to one of 18 different expanded detail forms [really 25, of which 7 have single-direction loci]. *Regime rays* are analytical representations for the significant *FS-rays* detail, through aggregation of same-oriented segmentals to single segments. The *regime rays* condense unknown detail re sequenced segmentals into analytical expressions which reveal then essence of that detail. Much of that analytical detail is can be found in a previous Abstract (Vetter 2007, accessible through CSEG: '2007 CSPG CSEG Joint Convention', Seismic Processing II).

**Orthorhombic Medium Simulation Example**

Figures 1 to 3 below show *regime rays* simulation detail for Schoenberg and Helbig's (1997)'orthorhombic standard model'. I have used Helbig's explicit Kelvin-Christoffel matrix expansion (1994, Appendix 4B, short version valid for orthorhombic and higher symmetry in medium natural coordinate frame), and a novel compact expression for vectored $\boldsymbol{r}_E$ , derived from the pathtime *ansatz* [(eqn 1b) in spherical coordinates ($\theta$, $\phi$ )], together with the pathtime minimizing derivatives re ($\theta$, $\phi$ ). The below equation 2 is its column-vectored form, after re-converting from spherical- back to rectangular coordinates.

As an important 'aside', eqn.2, columned as shown, is variously incorrect in the literature [e.g. Helbig 1994, p.13 eqn.1a.6{without common dispersion term}; Mensch and Rasolofosaon 1997 eqn.12; ... ]. Further, Auld's (1973) suggested 'carrier modulation' analogy, [velocity of carrier<--> vectored *phase velocity*, velocity of modulation envelop<-->vectored *group*- or *ray- velocity*] is not realistic/ applicable, nor are then '*phase*-' and '*group-velocity*' really relevant or proper designations for context of the *FS-ray*s *ansatz*.

Equation (3), also from eqn.1b plus derivatives, is important for visualization, transparency, and credibility of the *regime ray*s, which reveal pathtimes/ pathlengths/ velocity detail, and thus quantifiable heterogeneity.

$$t_{ON} = t_{OE} = (\boldsymbol{r}_E \bullet \boldsymbol{n}_N)/\ v_N\ (\boldsymbol{n}_N,\ c_{ij},\ \rho) = (x \sin\theta\cos\phi + y \sin\theta\sin\phi + z \cos\theta) /\ v_N\ (\theta, \phi\ ;\ c_{ij},\rho) \qquad (1a, 1b)$$

$$\begin{pmatrix} \frac{x}{t} \\ \frac{y}{t} \\ \frac{z}{t} \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} * v_N + \begin{pmatrix} 1-n_x^2 & -n_x n_y & -n_x n_z \\ -n_x n_y & 1-n_y^2 & -n_y n_z \\ -n_x n_z & -n_y n_z & 1-n_z^2 \end{pmatrix} \begin{pmatrix} \frac{\partial v_N}{\partial n_x} \\ \frac{\partial v_N}{\partial n_y} \\ \frac{\partial v_N}{\partial n_z} \end{pmatrix} \quad ; \quad \boldsymbol{n}_N = [\, n_x\ n_y\ n_z\,] \qquad (2)$$

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{z}{t}\right)^2 = v_E^{\,2} = v_N^{\,2} + \left(\frac{\partial v_N}{\partial\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial v_N}{\partial\phi}\right)^2 = v_N^{\,2} + \left(\sqrt{\left(\frac{\partial v_N}{\partial\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial v_N}{\partial\phi}\right)^2}\right)^2 \qquad (3)$$

Six *regime rays* are simulated for qP-, qSV- and qSH-modes, from origin to cross-diagonal between [*x y z*] = [3 0 0] → [0 3 3], for equi-angle stepped front-normals. This sampling shows changes from near-surface in-layers-dominant to deeper cross-layers and at-slant-encountered cracks. The displays are km-spatial, with progression SPEEDS made tangible through dot-density [40 intervals per second]; DENSE is slow and SPARSE is fast! The action per se is virtually on the energy flux loci from **O**rigin to **E** shown in RED, but detail is visualized through *regime ray* segments; fine-dots red link the **E**-points. Front-normal **N**-points and dotted *regime rays* are BLUE; fine-dots-blue link the **N**-points. The GREEN-framed **R**-**E**-**N** triangles are tangent plane portions anchored at **E**, with pythagoras **R**ight angle points linking to 2nd segments extended.

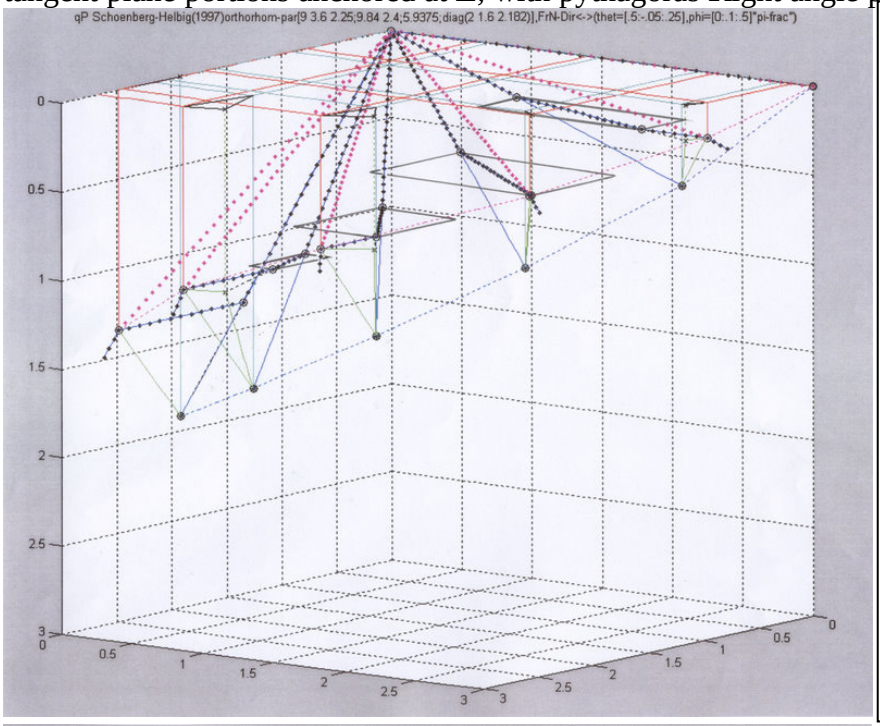


Fig.1: qP-mode *REGIME RAYS*
Schoenberg-Helbig orthorhombic model
Data for # 2 & # 4 ***FS-rays***

| # 2 | directions | *time* | *length* | *velo* |
|---|---|---|---|---|
| E[. 9627 . 2619 . 0674] | | 1. 000 | 2. 9670 | 2. 9670 |
| N[. 9393 . 3052 . 1564] | | .4328 | 1..2775 | 2. 9516 |
| Z[. 9511 . 3090 | 0 ] | .4192 | 1. 2526 | 2. 9884 |
| x[ 1. 0000 | 0 0 ] | 1. 1480 | . 4652 | 3. 1422 |
| vTA=2. 9953 | vRMS=2. 9959 | vPM=2. 9966 | | |
| lenPATH=2. 9953 | | vN=2. 9516 | | |
| P[ . 9574 . 2757 . 0860] | Polarization _____ | | | |

| # 4 | directions | *time* | *length* | *velo* |
|---|---|---|---|---|
| E[. 4817 . 5837 . 1577] | | 1. 000 | 2. 9101 | 2. 9101 |
| N[. 5237 . 7208 . 4540] | | . 5832 | 1. 6514 | 2. 8316 |
| Z[. 5878 . 8090 | 0 ] | . 2874 | . 9133 | 3. 1780 |
| y[ 0 1.0000 | 0 ] | . 1294 | . 5083 | 3. 9282 |
| vTA=3. 0730 | vRMS=3.0944 | vPM=3. 1159 | | |
| lenPATH=3. 0730 | | vN=2. 8316 | | |
| P[. 5010 . 8097 . 3056] | Polarization _____ | | | |

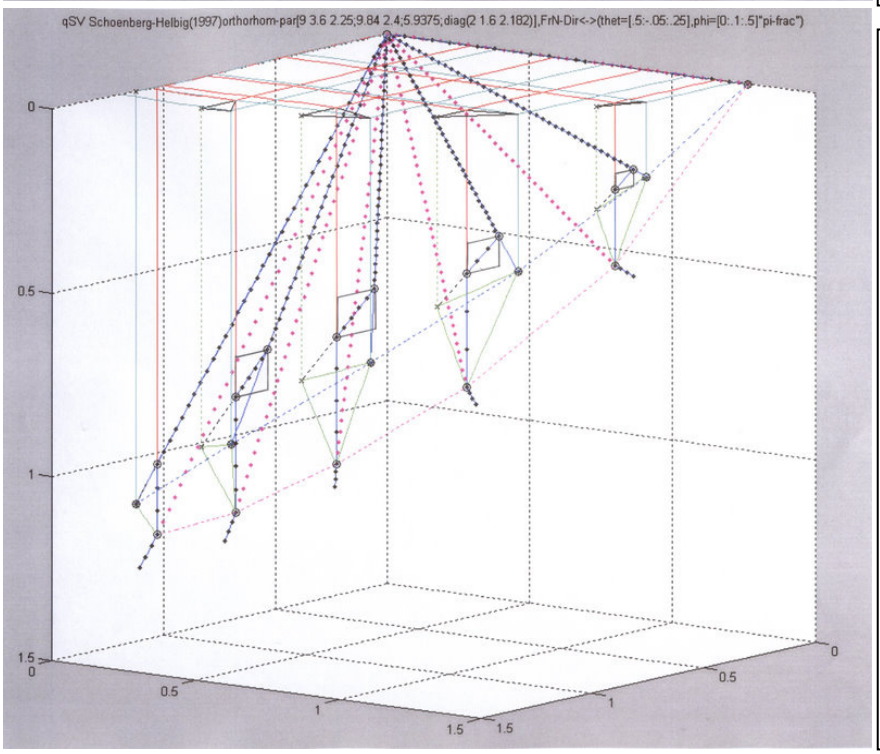


Fig.2: qSV-mode *REGIME RAYS*
Schoenberg-Helbig orthorhombic model
Data for # 2 & # 4 ***FS-rays***

| # 2 | directions | *time* | *length* | *velo* |
|---|---|---|---|---|
| E[. 8757 . 3485 . 3342] | | 1. 000 | 1. 3243 | 1. 3243 |
| N[. 9393 . 3052 . 1564] | | . 9501 | 1..2346 | 1. 2995 |
| X[ 0 . 3053 . 1564] | | . 0251 | . 0952 | 3. 7889 |
| z[ 0 0 1. 000] | | . 0248 | . 2061 | 8. 3067 |
| vTA=1. 5359 | vRMS=1. 9175 | vPM=2. 3939 | | |
| lenPATH=1.5359 | | vN=1. 2995 | | |
| P[-. 0652 -. 0838 . 9943] | Polarization_____ | | | |

| # 4 | directions | *time* | *length* | *velo* |
|---|---|---|---|---|
| E[. 3946 . 6566 . 6427] | | 1. 000 | 1. 5100 | 1. 5100 |
| N[. 5237 . 7208 . 4540] | | . 7753 | 1. 1377 | 1. 4674 |
| X[ 0 . 8462 . 5329] | | . 1176 | . 2025 | 1. 7225 |
| z[ 0 0 1.0000] | | . 1071 | . 3460 | 3. 2322 |
| vTA=1. 6863 | vRMS=1.7711 | vPM=1. 8602 | | |
| lenPATH=1. 6863 | | vN=1. 4674 | | |
| P[-. 2033 -. 2331 . 9510] | Polarization _____ | | | |

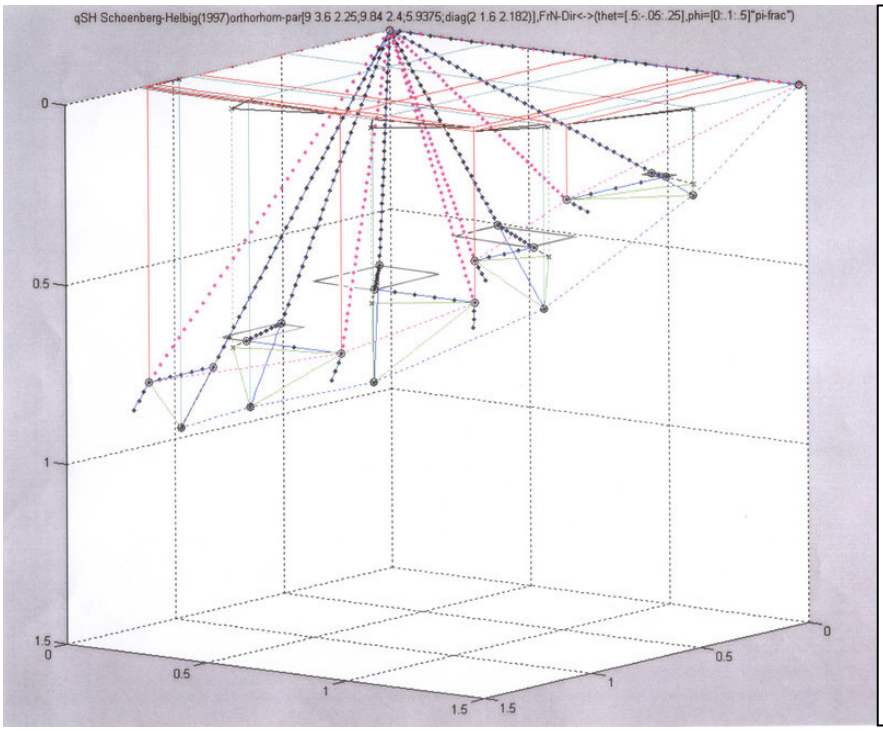


Fig.3: qSH-mode *REGIME RAYS*
Schoenberg-Helbig orthorhombic model

Data for # 2 & # 4 ***FS-rays***

| # 2 | directions | *time* | *length* | *velo* |
|---|---|---|---|---|
| E[. 8236 . 5520 . 1302] | | 1. 000 | 1. 6154 | 1. 6154 |
| N[. 9393 . 3052 . 1564] | | . 8647 | 1..3445 | 1. 5548 |
| Z[. 9511 . 3090 | 0 ] | ..0451 | . 0711 | 1. 5742 |
| y[ 0 | 1.0000 0 ] | . 0902 | . 4595 | 5. 0942 |
| vTA=1. 8749 | vRMS=2. 1314 | vPM=2. 4229 | | |
| lenPATH=1. 8749 | | vN=1. 5548 | | |
| P[-. 2814 . 9576 . 0623] | Polarization_____ | | | |

| # 4 | directions | *time* | *length* | *velo* |
|---|---|---|---|---|
| E[. 6945 . 6568 . .2938] | | 1. 000 | 1. 6571 | 1. 6571 |
| N[. 5237 . 7208 . 4540] | | . 6668 | 1. 0724 | 1. 6083 |
| Z[. 5878 . 8090 | 0 ] | . 2160 | . 3898 | 1. 8050 |
| x[ 1.0000 | 0 0 ] | . 1172 | . 3601 | 3. 0709 |
| vTA=1. 8222 | vRMS=1.8799 | vPM=1. 9393 | | |
| lenPATH=1. 8222 | | vN=1. 6083 | | |
| P[-. 8412 . 5386 -. 0479] | Polarization_____ | | | |

Focus initially on Fig.3 qSH-mode with its #2 *regime ray* and the boxed data. E-rowed underlined data is for red-dotted energy flux locus, so-called *group*- or *ray*-(velocity) representation. {N Z y} is code for regime ray segment orientations, here N for 3D front-normal, Z for in parallel to coordinate frame z-plane, and y for y-axis parallel. Segment 2 looks like a blur, but details the boundary of small z-plane rectangle with segment progressing diagonally; c.f. outlined plane areas for second segments of other *regime rays*. Note regime changed to {N Z x} for #4 *regime ray*, which through nearness of #3 and #4 energy flux loci suggests a 'flip-point', possibly even a cusping-like swerving. Attentive readers will notice and ponder the distinctly different *regime rays* patterns for the different modes; e.g. shear-modes *horizontal*/ *vertical* have second and third *regime ray* segments so-oriented. And remarkably FAST can manifest for third segments!

## Conclusions

*Regime rays* encapsulate and quantify the fine structured anisotropic medium heterogeneity encountered along wave disturbance energy-transport channels through their significant direction and propagation speed detail. Because that detail links to parameters of relevant elastic medium models, they will be important for seismic wave propagation-, and particularly for anisotropy velocity field- modeling, as also potentially for data to-medium-models inversing. I expect *regime rays* will bring anisotropy kinematics back into the *FS-ray theory* purview, however with minor refinement of its premises.

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